Lecture 22: Lévy Distributions

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1 Central Limit Theorem (from previously)

Let \( \{x_i\} \) be IID random variables with finite mean \( \mu \) and finite variance \( \sigma^2 \). Then the RV

\[
S_n = \frac{1}{\sigma \sqrt{n}} \left( \sum_{j=1}^{n} x_j - n \mu \right)
\]

converges to a standard \( N(0,1) \) Gaussian density.

2 Introduction

This lecture covers similar material to Lectures 12 and 13 in the 2003 lecture notes. Also, see Hughes\(^1\) §4.3.

Motivating question: What are the limiting distributions for a sum of IID variables? (And, in particular, those who violate the assumptions of the CLT.)

3 Lévy stability laws

Definitions: \( \{x_i\} \) RVs and \( X_N = \sum_{n=1}^{N} x_n \)

Characteristic function:

\[
\hat{p}_n (k) = [\hat{p}(k)]^n \\
\hat{p}_{n \times m} (k) = [\hat{p}_m (k)]^n = [\hat{p}(k)]^{n \times m}
\]

Introduce new variables \( Z_N = \frac{X_N}{a_N} \). The PDF of the \( Z_N \)'s is given by \( \hat{F}_N (x) = \frac{x}{a_N} \frac{1}{a_N} \).

\[
\hat{F}_N (a_N k) = \hat{p}_N (k)
\]

Substituting in,

\[
\hat{F}_{n \times m} (a_{nm} k) = \left[ \hat{F}_m (a_m k) \right]^n
\]

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As \( n \to \infty \), we want \( \hat{F}_n(k) \) to tend to some fixed distribution \( \hat{F}(k) \). Say that
\[
\lim_{m \to \infty} \frac{a_m}{a_n} = c_n,
\]

\[
\hat{F}(kc_n) = \left[ \hat{F}(k) \right]^n
\]  

(6)

Suppose I want to find
\[
\hat{F}(k\mu(\lambda)) = \left[ \hat{F}(k) \right]^\lambda
\]  

(7)

Introduce \( \psi(k) = \log \hat{F}(k) \)
\[
\psi(k\mu(\lambda)) = \lambda \psi(k)
\]  

(8)

Differentiate:
\[
k\mu'(\lambda) \psi'(k\mu(\lambda)) = \psi(\lambda)
\]  

(9)

\[
\mu(\lambda = 1) = 1
\]  

(10)

\[
k\mu'(1) \psi'(k) = \psi(k)
\]  

(11)

\[
\frac{d\psi}{dk} = \frac{\psi}{\mu'(1)k}
\]  

(12)

This has a solution:
\[
\psi(k) = \begin{cases} 
  v_1 |k|^\alpha & k > 0 \\
  v_2 |k|^\alpha & k < 0 
\end{cases}
\]  

(13)

\[
\hat{F}(k) = \begin{cases} 
  \exp(v_1 |k|^\alpha) & k > 0 \\
  \exp(v_2 |k|^\alpha) & k < 0 
\end{cases}
\]  

(14)

\[
v_1, v_2 \in \mathbb{C}
\]  

(15)

Thus we have this condition:
\[
\hat{F}(-k) = \hat{F}(k)^* 
\]  

(16)

\[
\implies \hat{F}(k) = \exp((c_1 + ic_2 \text{sgn}(k)) |k|^\alpha)
\]  

(17)

\[
c_1, c_2 \in \mathbb{R}
\]  

(18)

We can also write:
\[
\hat{F}(k) = \exp\left[-a |k|^\alpha \left(1 - i\beta \tan \frac{\alpha \pi}{2} \text{sgn}(k)\right)\right]
\]  

(19)

In this equation, \( \beta \) relates to the skewness of the distribution.

4 Lévy distribution

For above \( \hat{F} \) when \( \beta = 0 \) (ie. symmetric limiting distribution), you get the Lévy distribution. It is defined in terms of its Fourier transform.

\[
\hat{L}_\alpha(a, k) = \hat{F}(k) = \exp(-a |k|^\alpha)
\]  

(20)

Two special cases of the the Lévy distribution are \( \alpha = 1, 2 \).

\[
\begin{align*}
\alpha = 2 & \quad \hat{F}(k) = \exp(-ak^2) \quad \text{like Gaussian but more general} \\
\alpha = 1 & \quad \hat{F}(k) = \exp(-a |k|) \quad \text{Cauchy}
\end{align*}
\]  

(21)

(22)
4.1 Large \( x \) expansions of the Lévy distribution

\[
L_\alpha (a, x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp ikx - a |k|^\alpha dk
\]  
\[
= \frac{1}{\pi} \int_{0}^{\infty} \cos kx \exp -ak^\alpha dk
\]  
\[23\]
\[24\]
Integrate by parts:

\[
= \frac{a\alpha}{\pi} \int_{0}^{\infty} \sin \frac{k|x|}{|x|} \exp -ak^\alpha dk
= \frac{a\alpha}{\pi |x|^{1+\alpha}} \int_{0}^{\infty} \xi^{\alpha-1} \sin \xi \exp -\frac{a\xi^\alpha}{|x|^\alpha} d\xi
\]  
\[25\]
\[26\]
Where we have made the following substitutions:

\[\xi = k |x| , d\xi = dk |x|\]  
\[27\]
Taking the limit \( x \to \infty \):

\[
\sim_{x \to \infty} \frac{a\alpha}{\pi |x|^{1+\alpha}} \int_{0}^{\infty} \xi^{\alpha-1} \sin \xi d\xi
\]  
\[28\]
\[
\sim \frac{a\alpha \Gamma(\alpha) \sin \frac{\pi \alpha}{2}}{\pi |x|^{1+\alpha}} \text{ power-law tail}
\]  
\[29\]

4.2 Small \( x \) expansion

\[
L_\alpha (a, x) = \frac{1}{\pi} \int_{0}^{\infty} \cos kx \exp -ak^\alpha dk
\]  
\[30\]

Taylor expand cosine:

\[
= \frac{1}{\pi} \int_{0}^{\infty} \sum_{n=0}^{\infty} (-1)^n \frac{(kx)^{2n}}{(2n)!} \exp -ak^\alpha dk
\]  
\[31\]

Substitute: \( w = ak^\alpha \) \( dw = aak^{\alpha-1}dk \)

\[
= \frac{1}{\pi} \sum_{n=0}^{\infty} \int_{0}^{\infty} (-1)^n \frac{(x)^{2n}}{(2n)!} \frac{w^{2n/\alpha}}{a^{2n/\alpha}} \exp -w \frac{dw}{a^{2n/\alpha}}
\]  
\[32\]
\[33\]
We’re only interested in evaluating this equation as \( x \to 0 \):

\[
= \frac{1}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \frac{(x)^{2n}}{a^{2n/\alpha}} \Gamma \left( \frac{2n+1}{\alpha} \right)
\]  
\[34\]
\[
\sim \frac{1}{\pi} \frac{\Gamma \left( \frac{2}{\alpha} \right)}{a^{2/\alpha}} \text{ increases as } \alpha \to 0
\]  
\[35\]

5 Analogy with Central Limit Theorem?

5.1 Gnedenko-Doblin Theorem

\[
\frac{1}{a_N} p_N \mathbf{x}_{a_N} \to L_{\alpha, \beta} (a, x) \text{ iff the CDF } P(x) \text{ satisfies}
\]

1. \( \lim_{x \to \infty} \frac{p(-x)}{a - p(x)} = \frac{1-\beta}{1+\beta} \)

This is akin to the “amount” of probability on one side of the tail.
2. \[ \lim_{x \to \infty} \frac{1-p(x) + p(-x)}{1-p(x) + p(-x)} = r^\alpha \quad \forall r \]

We can think of this as saying how fast it is decaying.

5.2 Theorem

The distribution \( p(x) \) is in the basin of attraction of \( L_{\alpha,\beta}(a, x) \) if

\[
p(x) \sim \frac{A_+}{|x|^{1+\alpha}} \quad x \to \pm \infty
\]

\[
\beta = \frac{A_+ - A_-}{A_+ + A_-}
\]

If \( p(x) \) has power-law tails, then the sum of RVs goes to a Lévy distribution.

5.3 Scaling and superdiffusivity

Suppose that \( \{x_i\} \) follow a \( L_\alpha(a, x) \) distribution, then \( \hat{p}(k) = \exp -a \, |k|^\alpha \) and then the characteristic function at \( x_N = \sum_{n=1}^{N} x_n \) is given by

\[
\hat{p}_N(k) = [\exp -a \, |k|^\alpha]^N = \exp -a \, |k|^\alpha \, N
\]

\[
p_N(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp ikx - aN \, |k|^\alpha \, dk
\]

Now we substitute:

\[
k = VN^{-1/\alpha}, \quad dk = -dVN^{-1/\alpha}
\]

\[
p_N(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp iV \left( \frac{x}{N^{1/\alpha}} \right) - a \, |V|^{\alpha} \frac{dV}{N^{1/\alpha}} = \frac{1}{N^{1/\alpha}} L_\alpha \left( a, \frac{x}{N^{1/\alpha}} \right)
\]

Thus, the width scales like \( N^{1/\alpha} \).

If \( \alpha > 2 \), we get scaling that exceeds “square-root” scaling, i.e. a superdiffusive process.