Problem Set Number 10, 18.385j/2.036j
MIT (Fall 2014)

Rodolfo R. Rosales (MIT, Math. Dept., Cambridge, MA 02139)

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1 Problem 09.06.02 - Strogatz. Pecora and Carroll’s approach

Statement for problem 09.06.02

Pecora and Carroll’s approach for signal transmission/reception using the Lorenz system. In the pioneering work of Pecora and Carroll \(^1\) one of the receiver variables is simply set \textit{equal} to the corresponding transmitter variable. For instance, if \(x(t)\) is used as the transmitter drive signal, then the receiver equations are

\[
\begin{align*}
    x_r(t) & \equiv x(t), \\
    \frac{dy_r}{dt} & = r x(t) - y_r - x(t) z_r, \\
    \frac{dz_r}{dt} & = x(t) y_r - b z_r,
\end{align*}
\]

where the first equation is \textbf{not} a differential equation.\(^2\) Their numerical simulations, and a heuristic argument, suggested that \(y_r(t) \rightarrow y(t)\) and \(z_r(t) \rightarrow z(t)\) as \(t \rightarrow \infty\), even if there were differences in the initial conditions.

Here are the steps for simple proof of the result stated above, due to He and Vaidya.\(^3\)

\textbf{A. Show} that the error dynamics are governed by:

\[
\begin{align*}
    e_x(t) & \equiv 0, \\
    \frac{de_y}{dt} & = -e_y - x(t) e_z, \\
    \frac{de_z}{dt} & = x(t) e_y - b e_z,
\end{align*}
\]


\(^2\) This equation replaces the first equation \(\dot{x}_r = a (y_r - x_r)\) in a Lorenz system for \((x_r, y_r, z_r)\). Then \(x\) is used to replace \(x_r\) in the other two equations. The Lorenz system constants are \(\sigma, r, b\).

where \( e_x = x - x_r, \ e_y = y - y_r, \) and \( e_z = z - z_r. \)

**B. Show** that \( V = (e_y)^2 + (e_z)^2 \) is a Liapunov function.

**C. What do you conclude?**

### 2 Hill equation problem #04 (with damping)

**Statement:** Hill equation problem #04 (with damping)

Let \( S = S(\xi) \) be a periodic (of period \( 2\pi \)) function — i.e.: \( S(\xi + 2\pi) = S(\xi). \) Consider now the damped Hill equation problem

\[
\ddot{x} + 2\nu \dot{x} + (k^2 + a^2 S(\omega t)) \ x = 0, \tag{2.1}
\]

where \( \nu, \ k, \ a, \ \omega > 0 \) are constants — note that the coefficients period is \( T = \frac{2\pi}{\omega}. \)

**Problem tasks:**

1. Write the equations in the standard form \( \dot{X} = A(\omega t) \ X, \) where \( A \) is a \( 2 \times 2 \) matrix with period \( 2\pi \) and \( X \) is a two-vector.

2. Write the Floquet multipliers \( \lambda_j \) in terms of \( \alpha = \frac{1}{2} \text{Tr}(R), \) where \( R \) is the Floquet matrix.
   
   Hint. \( \Delta = \det(R) \) can be computed explicitly.

3. Write the stability/instability condition in terms of \( \alpha. \)

4. Find the function \( \alpha_0 = \lim_{a \to 0} \alpha. \) Then use it to identify the places, if any, where an instability may occur for \( 0 < a \ll 1. \) That is, the values \( k = k_* \) such that, for \( 0 < a \ll 1, \) instabilities can arise for \( k \) near \( k_* \) only.
   
   Hint. For a small instabilities only arise near \( k \)'s where a Floquet multiplier satisfies \( |\lambda_j| = 1 \) for \( a = 0. \)

5. Plot \( \alpha_0 \) versus \( k/\omega, \) with \( \nu/\omega \) fixed, in a graph that includes the neutral stability curves. Use the range \( 0 \leq k/\omega \leq 5.1 \) and take \( \nu/\omega = 0.06, 0.20, 0.50. \)

   The neutral stability curves are lines in the \( \alpha-k \) plane such that: a Floquet multiplier satisfies \( |\lambda_j| = 1 \) when/where the graph of \( \alpha \) intersects the curve. You should know these curves from item 3.

   Hint: when solving item 4 you should find that \( \alpha_0 \) is a function of \( k/\omega \) and \( \nu/\omega \) only, while the neutral stability boundary depends on \( \nu/\omega \) only.

6. Plot \( \alpha_0 \) versus \( k/\omega, \) with \( \nu \) a function of \( k, \) in a graph that includes the neutral stability curves. Use the range \( 0 \leq k/\omega \leq 5.1 \) and take \( \nu = 0.06 \ k, 0.12 \ k, 0.10 \frac{k^2}{\omega}. \)

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**THE END.**