Linear Programming

We now start with the study of smoothed analysis for linear programming. In the forthcoming lectures we shall cover three methods for LP:

1.- Elementary/Naive/Stupid methods (this lecture).
2.- Simplex method (part in this lecture).
3.- Interior point method.

Introduction

Given $a_1, \ldots, a_n \in \mathbb{R}^d$ and $c \in \mathbb{R}^d$, Dan’s favorite linear program is the following

\[
\begin{align*}
\text{[LP1]} \quad \max \quad & \alpha \\
\text{s.t.} \quad & ac \in \text{ch}(a_1, \ldots, a_n), \\
& \alpha \geq 0.
\end{align*}
\]

where $\text{ch}(a_1, \ldots, a_n)$ denotes the convex hull defined by the vectors $a_1, \ldots, a_n$. We know that $c \in \text{ch}(a_1, \ldots, a_n)$ if and only if there exists $y_1, \ldots, y_n \geq 0$ such that $\sum_{i=1}^n y_i = 1$ and $c = \sum_{i=1}^n y_i a_i$. Hence we can write [LP1] in an equivalent form

\[
\begin{align*}
\text{[LP2]} \quad \max \quad & \alpha \\
\text{s.t.} \quad & \exists y_1, \ldots, y_n \geq 0 \\
& \sum_{i=1}^n y_i = 1, \sum_{i=1}^n y_i a_i = ac.
\end{align*}
\]
By letting $y'_i = \frac{c_i}{\alpha}$, we have that $y'_i \geq 0$, $\sum_{i=1}^n y'_i = \frac{1}{\alpha}$, and $c = \sum_{i=1}^n y'_i a_i$. Then, maximizing $\alpha$ is the same as minimizing $\sum_{i=1}^n y'_{i'}$ and hence [LP2] becomes

$$\left[ \text{LP3} \right] \text{ min } \sum_{i=1}^n y'_i$$

s.t. $\sum_{i=1}^n y'_i a_i = c,$

$y'_i \geq 0.$

1.- Von Neumann’s algorithm for LP

We know see an elementary algorithm to solve [LP1]. The algorithm will use binary search on $\alpha$. And it will call a subroutine which decides whether a given $\alpha c$ belongs to $\text{ch}(a_1, \ldots, a_n)$, this is called the decision problem.

To solve the decision problem Von Neumann’s algorithm proceeds as follows:

- Take $x_0 \in \text{ch}(a_1, \ldots, a_n)$, say $x_0 = \frac{1}{n} \sum_{i=1}^n a_i$.
- Choose $i$ maximizing $\langle (\alpha c - x_0), (a_i - x_0) \rangle$ ($i = 2$ in the figure below).
- Find the point $x$ from $x_0$ towards $a_i$ (i.e. $x \in \text{ch}(x_0, a_i)$) closest to $\alpha c$.
- $x_0 = x$ and repeat.

This procedure converges since whenever $\alpha c \in \text{ch}(a_1, \ldots, a_n)$ we have $\max_i \{\langle (\alpha c - x_0), (a_i - x_0) \rangle\} \geq 0$ for some $i$. Otherwise, $\langle (\alpha c - x_0), (a_i - x_0) \rangle < 0$ for all $a_i$. This implies that there is an hyperplane separating $x_0$ from $\{a_1, \ldots, a_n\}$, from where $\alpha c \not\in \text{ch}(a_1, \ldots, a_n)$.
Theorem 1 (Dantzig). Von Neumann’s Algorithm obtains a point \( x_0 \) such that \( \|x_0 - \alpha c\| < \epsilon \) in

\[
\frac{4 \max\{||\alpha c||, \max_i ||a_i||\}}{\epsilon^2}
\]

iterations.

Theorem 2 (Freund-Epelman). Von Neumann’s Algorithm obtains a point \( x_0 \) such that \( ||x_0 - \alpha c|| < \epsilon \) in

\[
\frac{8 \max\{||\alpha c||, \max_i ||a_i||\} \log \left( \frac{||x_0 - \alpha c||}{\epsilon} \right)}{r^2}
\]

iterations. Where \( r = \text{distance}(||\alpha c||, \text{boundary}(\text{ch}(a_1, \ldots a_n))) \). \( r \) is a condition number of the linear program.

2.- The Simplex method

Consider again the linear program [LP1]. A basic feasible solution of [LP1] is collection of \( d \) points, \( B \subset \{a_1, \ldots, a_n\} \ (|B| = d) \) such that \( \alpha c \in \text{ch}(B) \) for some \( \alpha \geq 0 \).

![Simplex method diagram](image)

The simplex method proceeds as follows:

- Find a Basic feasible solution \( B \).
- Find point, say \( a \), in \( \{a_1, \ldots, a_n\} \) above (with respect to \( c \)) the hyperplane ch\((B)\).
- Remove one point, \( b \), from \( B \cup \{a\} \) so that \( B' = B \cup \{a\} \setminus \{b\} \) is a basic feasible solution.
- \( B = B \cup \{a\} \setminus \{b\} \) and repeat.
**Initialization:** Plant a Basic feasible solution, i.e. put $d$ points very close to the origin (say at distance $\epsilon$), so that they are not involved in an optimal solution. This is also known as the big $M$ method since, $\min\{\sum_{i=1}^{n} y_i : \sum_{i=1}^{n} y_i a_i = c, y_i \geq 0\}$ can be written as

$$[\text{LP3}] \min \quad \sum_{i=1}^{n} y_i + M \sum_{j=1}^{d} z_j$$

s.t. 
$$\sum_{i=1}^{n} y_i a_i + \sum_{j=1}^{d} z_j e_j = c,$$
$$y_i \geq 0.$$

Which again, by letting $M = 1/\epsilon$, is equivalent to

$$[\text{LP3}] \min \quad \sum_{i=1}^{n} y_i + \sum_{j=1}^{d} z_j$$

s.t. 
$$\sum_{i=1}^{n} y_i a_i + \sum_{j=1}^{d} z_j (\epsilon e_j) = c,$$
$$y_i \geq 0,$$

and this is exactly the initialization method described above.

**Duality:** Another way of looking at Dan’s linear program [LP1] is the following. Find a plane $H$ and the minimum $\alpha$ such that, $\alpha c \in H$ and $\{a_1, \ldots, a_n\}$ are beneath $H$.

Since $H_x = \{a : \langle a, x \rangle = 1\}$ is the plane normal to a given vector $x$. The above problem,
know as the \textit{dual}, can be stated as

\begin{align*}
\text{[Dual-LP1]} \quad & \text{min} \quad \alpha \\
\text{subject to} \quad & \langle \alpha c, x \rangle = 1 \\
& \langle a_i, x \rangle \leq 1 \text{ for all } i.
\end{align*}

Now $\langle c, x \rangle = 1/\alpha$, hence the problem is simply

\begin{align*}
\text{[Dual-LP3]} \quad & \text{max} \quad \langle c, x \rangle \\
\text{subject to} \quad & \langle a_i, x \rangle \leq 1 \text{ for all } i.
\end{align*}

We can conclude the following result.

\textbf{Theorem 3.} \textit{If primal [LP1] has a solution, then the dual [Dual-LP1] has the same solution.}