Hint for Problem 2(b)

You may want to use the following fact. Let

$$X = \sum_{i=1}^{k} x_i.$$

Then

$$k \sum_{i=1}^{k} x_i^2 \geq X^2.$$

Hint for Problem 2(c)

Under the given embedding, we get

$$L_{K_n} = \sum_{u<v} L_{u,v}.$$

For each edge, use the inequality derived in part (b) to show

$$L_{u,v} \preceq (v-u) \sum_{i=u}^{v-1} L_{i,i+1}$$

and plug it into the summation above.

Hint for Problem 2(d)

Let $T_n(u, v)$ denote the path in the tree from u to v. It has length at most $2 \log n$. What does this give when we apply our path inequality to

$$L_{K_n} = \sum_{u<v} L_{u,v}$$

under this embedding? What is the largest number of times any edge of the tree can appear in the paths in the sum?

Hint for Problem 3

Let P_n be the path graph on n vertices, and let T_m be the complete binary tree on m vertices. Consider the product graph $G = P_n \times T_m$.

- What is the smallest nontrivial eigenvalue of L_G? How does it depend on n and m?
- If n is much bigger than m, what does the cut coming from λ_2 look like? And what is its conductance?
- What about if m is much bigger than n?

Hint for Problem 4(b)

Apply the Markov inequality to

$$e^X = \prod_i e^{x_i}.$$

Hint for Problem 6(a)

Argue that it suffices to show that there is no codeword with less than αN ones. Let S be the set of these nonzero bits. What does it mean if a vertex on the right has exactly one neighbor in S? Can you show that this always occurs?

Hint for Problem 6(c)

Reduce to the case where the nearest codeword at the beginning is the zero vector. If you produce a keyword with more than $\alpha N/2$ nonzeros, how many vertices on the right can you show have unique neighbors on the left? What can you use this to say about the number of violated constraints?

Hint for Problem 7(d)

Let

$$c = \frac{d - \mu_n}{n},$$

and consider the matrix $B = A - c\mathbf{1}\mathbf{1}^T$, where $\mathbf{1}$ is the all-ones vector. What are the eigenvalues of B? Find a way to relate these to the size of an independent set using Courant-Fischer and the right test vector.