1 Outline

Last time, we proved the Brunn-Minkowski inequality for boxes. Today we’ll go over the general version of the Brunn-Minkowski inequality and then move on to applications, including the Isoperimetric inequality and Grunbaum’s theorem.

2 The Brunn-Minkowski inequality

**Theorem 1** Let \( A, B \subseteq \mathbb{R}^n \) be compact measurable sets. Then

\[
(\text{Vol}(A \oplus B))^{1/n} \geq (\text{Vol}(A))^{1/n} + (\text{Vol}(B))^{1/n}.
\]

The equality holds when \( A \) is a translation of a dilation of \( B \) (up to zero-measure sets).

**Proof** An equivalent version of Brunn-Minkowski inequality is given by

\[
\left( \text{Vol}(\lambda A \oplus (1 - \lambda)B) \right)^{1/n} \geq \lambda(\text{Vol}(A))^{1/n} + (1 - \lambda)(\text{Vol}(B))^{1/n}, \; \forall \lambda \in [0,1].
\]

The equivalence of (1) and (2) follows from the fact that \( \text{Vol}(\lambda A) = \lambda^n \text{Vol}(A) \):

\[
\left( \text{Vol}(\lambda A \oplus (1 - \lambda)B) \right)^{1/n} \geq \left( \text{Vol}(\lambda A) \right)^{1/n} + \left( \text{Vol}((1 - \lambda)B) \right)^{1/n}
= (\lambda^n \text{Vol}(A))^{1/n} + ((1 - \lambda)^n \text{Vol}(B))^{1/n}
= \lambda(\text{Vol}(A))^{1/n} + (1 - \lambda)(\text{Vol}(B))^{1/n}.
\]

The inequality (2) implies that the \( n^{th} \) root of the volume function is concave with respect to the Minkowski sum.

Here, we sketch the proof for Theorem 1 by proving (1) for any set constructed from a finite collection of boxes. The proof can be generalized to any measurable set by approximating the set with a sequence of finite collections of boxes and taking the limit. We omit the analysis details here.

Let \( A \) and \( B \) be finite collections of boxes in \( \mathbb{R}^n \). We prove (1) by induction on the number of boxes in \( A \cup B \). Define the following subsets of \( \mathbb{R}^n \):

\[
A^+ = A \cap \{ x \in \mathbb{R}^n | x_n \geq 0 \}, \; A^- = A \cap \{ x \in \mathbb{R}^n | x_n \leq 0 \},
B^+ = B \cap \{ x \in \mathbb{R}^n | x_n \geq 0 \}, \; B^- = B \cap \{ x \in \mathbb{R}^n | x_n \leq 0 \}.
\]

Translate \( A \) and \( B \) such that the following conditions hold:

1. \( A \) has some pair of boxes separated by the hyperplane \( \{ x \in \mathbb{R}^n | x_1 = 0 \} \), i.e. there exists a box that lies completely in the halfspace \( \{ x \in \mathbb{R}^n | x_1 \geq 0 \} \) and there is some other box that lies in its complement half-space (see figure 1). (If there’s no such box in that direction we can change coordinates.)

2. It holds that

\[
\frac{\text{Vol}(A^+)}{\text{Vol}(A)} = \frac{\text{Vol}(B^+)}{\text{Vol}(B)}.
\]
Note that translation of $A$ or $B$ just translates $A \oplus B$, so any statement about the translated sets holds for the original ones.

Since $A^+$ and $A^-$ are strict subsets of $A$, we know that $A^+ \cup B^+$ and $A^- \cup B^-$ have fewer boxes than $A \cup B$. Therefore, (1) is true for them by the induction hypothesis. Moreover, $A^+ \oplus B^+$ and $A^- \oplus B^-$ are disjoint because they differ in sign of the $x_1$ coordinate. Hence, we have

$$\text{Vol}(A \oplus B) \geq \text{Vol}(A^+ \oplus B^+) + \text{Vol}(A^- \oplus B^-)$$

$$\geq (\text{Vol}(A^+)^{1/n} + \text{Vol}(B^+)^{1/n})^n + (\text{Vol}(A^-)^{1/n} + \text{Vol}(B^-)^{1/n})^n$$

$$= \text{Vol}(A^+) \left(1 + \left(\frac{\text{Vol}(B^+)}{\text{Vol}(A^-)}\right)^{1/n}\right)^n + \text{Vol}(A^-) \left(1 + \left(\frac{\text{Vol}(B^-)}{\text{Vol}(A^+)}\right)^{1/n}\right)^n$$

$$= (\text{Vol}(A^+) + \text{Vol}(A^-)) \left(1 + \left(\frac{\text{Vol}(B)}{\text{Vol}(A)}\right)^{1/n}\right)^n$$

$$= (\text{Vol}(A)^{1/n} + \text{Vol}(B)^{1/n})^n,$$

where the second inequality follows from the induction hypothesis, and the second equality is implied by (5).

\[\text{Figure 1: } A^+ \text{ and } B^+ \text{ as defined in the proof of Theorem 1.}\]

### 3 Applications of Brunn-Minkowski Inequality

In this section, we demonstrate the power of Brunn-Minkowski inequality by using it to prove some important theorems in convex geometry.

#### 3.1 Volumes of Parallel Slices

Let $K \in \mathbb{R}^n$ be a convex body. A parallel slice, denoted by $K_t$, is defined as an intersection of the body with a hyperplane, i.e.

$$K_t = K \cap \{x \in \mathbb{R}^n| x_1 = t\}.$$

(7)
Define the volume of the parallel slice $K_t$, denoted by $v_K(t)$, to be its $(n - 1)$-dimensional volume.

\[v_K(t) = \text{Vol}_{n-1}(K_t).\] 

We are interested in the behavior of the function $v_K(t)$, and in particular, in whether it is concave.

Consider the Euclidean ball in $\mathbb{R}^n$. The following plots of $v_K(t)$ for different $n$ suggest that except for $n = 2$, the function $v_K(t)$ is not concave in $t$.

As another example, consider a circular cone in $\mathbb{R}^3$. The volume of a parallel slice is proportional to $t^2$, so $v_K(t)$ is not concave. More generally, $v_K(t)$ is proportional to $t^{n-1}$ for a circular cone in $\mathbb{R}^n$. This suggests that the $(n - 1)^{th}$ root of $v_K$ is a concave function. This guess is verified by Brunn’s theorem.

**Theorem 2** (Brunn’s Theorem) Let $K$ be a convex body, and let $v_K(t)$ be defined as in (8). Then the function $v_K(t)^{\frac{1}{n-1}}$ is concave.

**Proof** Let $s, r, t \in \mathbb{R}$ with $s = (1 - \lambda)r + \lambda t$ for some $\lambda \in [0,1]$. Define the $(n - 1)$-dimensional slices $K_r, K_s, K_t$ as in (7). First, we claim that

\[(1 - \lambda)A_r \oplus \lambda A_t \subseteq A_s.\] 

We show this by proving that for any $x \in A_r$, $y \in A_t$, we have $z = (1 - \lambda)x \oplus \lambda y \in A_s$, as follows. Connect the points $(r, x)$ and $(t, y)$ with a straight line (see figure 2). By convexity of $K$, the line lies completely in the body. In particular, the point $(s, z)$, which is a convex combination of $(r, x)$ and $(t, y)$, lies in $A_s$. Therefore, $z \in A_s$ and the claim in (9) is true. Now, by applying the version of Brunn-Minkowski inequality in (2), we have

\[
\text{Vol}(A_s)^{\frac{1}{n-1}} \geq (1 - \lambda)\text{Vol}(A_r)^{\frac{1}{n-1}} + \lambda \text{Vol}(A_t)^{\frac{1}{n-1}}
\]

\[
\Rightarrow v_K(s)^{\frac{1}{n-1}} \geq (1 - \lambda)v_K(r)^{\frac{1}{n-1}} + \lambda v_K(t)^{\frac{1}{n-1}}
\]

\[(10)\]

**3.2 Isoperimetric Inequality**

A few lectures ago, we asked the question of finding the body of a given volume with the smallest surface area. The answer, namely the Euclidean ball, is a direct consequence of the Isoperimetric inequality. Before stating the theorem, let us define the surface area of a body using the Minkowski sum.

**Definition 3** Let $K$ be a body. The surface area of $K$ is defined as the differential rate of volume increase as we add a small Euclidean ball to the body, i.e.,

\[S(K) = \text{Vol}(\partial K) = \lim_{\epsilon \to 0} \frac{\text{Vol}(K \oplus \epsilon B^n_2) - \text{Vol}(K)}{\epsilon}.\] 

\[(11)\]
Now we state the theorem:

**Theorem 4 (Isoperimetric inequality)** For any convex body $K$, with $n$-dimensional volume $V(K)$ and surface area $S(K)$,

$$\left( \frac{V(K)}{V(B^n_2)} \right)^{1/n} \leq \left( \frac{S(K)}{S(B^n_2)} \right)^{\frac{1}{n-1}}$$

(12)

**Proof** By applying the Brunn-Minkowski inequality, we have the following:

$$V(K \oplus \epsilon B^n_2) \geq [V(K)^{1/n} + \epsilon V(B^n_2)^{1/n}]^n$$

$$= V(K) \left[ 1 + \epsilon \left( \frac{V(B^n_2)}{V(K)} \right)^{1/n} \right]$$

$$\geq V(K) \left[ 1 + n \epsilon \left( \frac{V(B^n_2)}{V(K)} \right)^{1/n} \right]$$

(13)

where the second inequality is obtained by keeping the first two terms of the Taylor expansion of $(1 + x)^n$.

Now, the definition of surface area in (11) implies:

$$S(K) = V(\partial K) \geq V(K) + n V(K) \left( \frac{V(B^n_2)}{V(K)} \right)^{1/n} - V(K)$$

$$= n V(K) \left( \frac{V(B^n_2)}{V(K)} \right)^{1/n}$$

$$= n V(K) \frac{n-1}{n} V(B^n_2)^{1/n}.$$ (14)

For an $n$-dimensional unit ball, we have $S(B^n_2) = n V(B^n_2)$. Therefore,

$$\frac{S(K)}{S(B^n_2)} \geq \frac{n V(K) \frac{n-1}{n} V(B^n_2)^{1/n}}{n V(B^n_2)}$$

$$\Rightarrow \left( \frac{S(K)}{S(B^n_2)} \right)^{\frac{1}{n-1}} \geq \left( \frac{n V(K) \frac{n-1}{n} V(B^n_2)^{1/n}}{n V(B^n_2)} \right)^{\frac{1}{n-1}}$$

$$= \left( \frac{V(K)}{V(B^n_2)} \right)^{1/n}$$

(15)

[\square]

### 3.3 Grunbaum’s Theorem

Given a high-dimensional convex body, we would like to pick a point $x$ such that for any cut of the body by a hyperplane, the piece containing $x$ is big. A reasonable choice for $x$ is the centroid, i.e.

$$x = \frac{1}{\text{Vol}(K)} \int_{y \in K} y dy.$$  

This choice guarantees to get at least half of the volume for any origin symmetric body, such as a cube or a ball. The question is how much we are guaranteed to get for a general convex body, and in particular, what body gives the worst case. Do we get a constant fraction of the body, or does the guarantee depend on dimension?
Let us first consider the simple example of a circular \( n \)-dimensional cone (figure 3). Suppose we cut the cone \( C \) by the hyperplane \( \{ x_1 = \bar{x}_1 \} \) at its centroid, where

\[
\bar{x}_1 = \frac{1}{\text{Vol}(C)} \int_{t=0}^{h} t \cdot \text{Vol}_{n-1} \left( \frac{tR}{h} \right)^{n-1} dt = \frac{n}{n+1} h. \tag{16}
\]

Grunbaum’s theorem states that the circular cone is indeed the worst case if we choose the centroid.

\[ \text{Figure 3: } n\text{-dimensional circular cone.} \]

First we’ll need the following lemma:

**Lemma 5** Let \( L = C \cap \{ x_1 \leq \bar{x}_1 \} \) by the left side of the cone (which is \( x_1 \)-aligned with vertex at the origin). Then \( \frac{1}{2} \geq \frac{V(L)}{V(C)} \geq \frac{1}{e} \).

**Proof**

\[
\frac{V(L)}{V(C)} = \frac{V(\frac{n}{n+1} C)}{V(C)} = \left( \frac{n}{n+1} \right)^n \\
\frac{1}{2} \leq \left( \frac{n}{n+1} \right)^n \leq \frac{1}{e}
\]

\[ \blacksquare \]

**Theorem 6** (Grunbaum’s Theorem) Let \( K \) be a convex body, and divide it into \( K_1 \) and \( K_2 \) using a hyperplane. If \( K_1 \) contains the centroid of \( K \), then

\[ \frac{\text{Vol}(K_1)}{\text{Vol}(K)} \geq \frac{1}{e}. \tag{17} \]

In particular, the hyperplane through the centroid divides the volume into almost equal pieces, and the worst case ratio is approximately 0.37 : 0.63.

**Proof** WLOG, change coordinates with an affine transformation so that the centroid is the origin and the hyperplane \( H \) used to cut is \( x_1 = 0 \). Then perform the following operations:

1. Replace every \( (n-1) \)-dimensional slice \( K_t \) with an \( (n-1) \)-dimensional ball with the same volume to get \( K' \), which is convex per Lemma 7 below.

2. Turn \( K' \) into a cone, such that the ratio gets smaller per Lemma 8 below.

**Lemma 7** \( K' \) is convex.

**Proof** Let \( K'_t = K' \cap \{ x_1 = t \} \) be a parallel slice in the modified body. The radius of \( K'_t \) is proportional to \( V(K_t)^{\frac{1}{n-1}} \). By applying Brunn-Minkowski inequality, we get that \( V(K_t)^{\frac{1}{n-1}} \) is a concave function in \( t \). Thus \( K' \) is convex. \[ \blacksquare \]
Lemma 8 We can turn $K'$ into a cone while decreasing the ratio.

Proof Let $K'_+ = K' \cap \{x_1 \geq 0\}, K'_- = K' \cap \{x_1 \leq 0\}$. Make a cone $yQ_0$ by picking $y$ having $x_1$ coordinate positive on the $x_1$-axis, and $V(yQ_0) = V(K'_+).$ Extend the code in the $\{x_1 \leq 0\}$ region, so that the volume of the extended part equals $V(K'_-)$; name this code $C'$. Now by Lemma 5, the centroid of $C'$ must lie in $yQ_0$. Let $H'$ be the translation of $H$ along the $x_1$-axis so that it contains the centroid of $C'$. Then

$$r(K, H) = r(C', H) \geq r(C', H') \geq 1/e.$$ 

This completes the proof of Grunbaum’s theorem.

4 Next Time

Next time, we will discuss approximating the volume of a convex body.