To begin, let me point out that there was a typo in the lecture notes from last lecture. Lemma 6.4.1 should have said:

**Lemma 7.0.1.**

\[
\Pr[X_1 = a_1|Y_2Y_3 = b_2b_3] = \sum_{a_2: (a_1, a_2) \in C_{12}} \Pr[X_2 = a_2|X_1 = a_1] \Pr[X_2 = a_2|Y_2 = b_2] \Pr[X_2 = a_2|Y_3 = b_3].
\]

## 7.1 Markov Property

Last lecture, we considered three variables \(X_1, X_2\) and \(X_3\) chosen uniformly from those that satisfy \((X_1, X_2) \in C_{12} \subseteq A_1 \times A_2\) and \((X_2, X_3) \in C_{23} \subseteq A_2 \times A_3\). We claimed that the variables \((X_1, X_2, X_3)\) then satisfy what the book calls the “Markov” property. That is, for all \(a_1, a_2, a_3\),

\[
\Pr[X_1X_3 = a_1a_3|X_2 = a_2] = \Pr[X_1 = a_1|X_2 = a_2] \Pr[X_3 = a_3|X_2 = a_2].
\]

I’ll now sketch a proof. It basically follows by by recalling the definition of the probability of an even conditioned on \(X_2 = a_2\). We first note that the set of choices for \((X_1, X_2, X_3)\) given that \(X_2 = a_2\) is

\[
S_{a_2} \defeq \{(X_1, a_2, X_3) : (X_1, a_2) \in C_{12} \text{ and } (a_2, X_3) \in C_{23}\}.
\]

Conditioning on \(X_2 = a_2\), we obtain a sample chosen uniformly from \(S_{a_2}\). Thus, for \((a_1, a_2, a_3) \in S_{a_2}\),

\[
\Pr[(X_1, X_2, X_3) = (a_1, a_2, a_3)|X_2 = a_2] = \frac{1}{|S_{a_2}|}.
\]

Note that

\[
|S_{a_2}| = |\{a_1 : (a_1, a_2) \in C_{12}\}||\{a_3 : (a_2, a_3) \in C_{23}\}|
\]

The claim now follows from observing that

\[
\Pr[(X_1, X_2) = (a_1, a_2)|X_2 = a_2] = \frac{|\{a_3 : (a_2, a_3) \in C_{23}\}|}{|S_{a_2}|} = \frac{1}{|\{a_1 : (a_1, a_2) \in C_{12}\}|}
\]

and

\[
\Pr[(X_2, X_3) = (a_2, a_3)|X_2 = a_2] = \frac{|\{a_1 : (a_1, a_2) \in C_{12}\}|}{|S_{a_2}|} = \frac{1}{|\{a_3 : (a_2, a_3) \in C_{23}\}|}.
\]
7.2 Simplifying Probability Computation

First, let's establish that the fundamental quantities we are interested in have the form

\[ P [X_i = a_i | E], \]

where \( E \) is some event, usually a union of the observed variables. We will typically want these values for all \( a_i \), so we really want a vector

\[ (P [X_i = a_1 | E], P [X_i = a_2 | E], \ldots, P [X_i = a_n | E]), \]

where \( a_1, \ldots, a_n \) are the symbols in the alphabet \( A_i \). We will denote such a vector by

\[ \vec{P} [X_i | E]. \]

Using this notation, and letting \( \odot \) denote componentwise product \( ((a, b) \odot (c, d) = (ac, bd)) \), we have

\[ \vec{P}^{post} [X_i | E] = \vec{P}^{prior} [X_i] \odot \vec{P}^{ext} [X_i | E] \]

Before returning to our probability computations for \((X_1, X_2, X_3)\), I'll also introduce the simpler notation \( P^{ext} [X_i = a_i | Y_i] \) for \( P^{ext} [X_i = a_i | Y_i = b_i] \). We will use this notation whenever \( b_i \) is fixed throughout our computation, which it generally is as it is what was received.

We then have, from Lemma 6.2.1,

\[ \vec{P}^{post} [X_1 | Y_1 Y_2 Y_3] = \vec{P}^{prior} [X_1] \odot \vec{P}^{ext} [X_1 | Y_1] \odot \vec{P}^{ext} [X_1 | Y_2 Y_3], \]

and, from Lemma 7.0.1,

\[ P^{ext} [X_1 = a_1 | Y_2 Y_3] = \sum_{a_2: (a_1, a_2) \in C_{12}} P [X_2 = a_2 | X_1 = a_1] \vec{P}^{ext} [X_2 = a_2 | Y_2] \vec{P}^{ext} [X_2 = a_2 | Y_3]. \]

Using these formulas, we go through the following steps to compute \( \vec{P}^{post} [X_1 | Y_1 Y_2 Y_3] \).

1. Compute \( \vec{P}^{ext} [X_2 | Y_3] \). This computation only depends upon \( Y_3 \), and comes from the formula:
   \[ P^{ext} [X_2 = a_2 | Y_3] \sim \sum_{a_3: (a_2, a_3) \in C_{23}} P [X_3 = a_3 | X_2 = a_2] P [Y_3 | X_3 = a_3]. \]

2. Compute \( \vec{P}^{ext} [X_2 = | Y_2] \), and compute
   \[ \vec{P}^{ext} [X_2 = | Y_2] \odot P^{ext} [X_2 = a_2 | Y_3]. \]

3. For each \( a_1 \), compute
   \[ \vec{P} [X_2 | X_1 = a_1] \odot \vec{P}^{ext} [X_2 | Y_2] \odot \vec{P}^{ext} [X_2 | Y_3], \]
   and then sum the resulting vector.

4. Take the output of the previous step, and \( \odot \) product it with \( \vec{P}^{prior} [X_1] \odot \vec{P}^{ext} [X_1 | Y_1]. \)

If you look at the flow of this computation, it can be understood as a vector being passed from \( X_3 \) to \( X_2 \) between steps 1 and 2, and a vector begin passed from \( X_2 \) to \( X_1 \) between steps 3 and 4.