1. A meat packing plant produces 480 hams, 400 pork bellies and 230 picnic hams every day; each of these products can be sold either fresh or smoked. The total number of hams, bellies and picnics that can be smoked during a normal working day is 420; in addition, up to 250 products can be smoked on overtime at a higher cost. The net profits are as follows:

<table>
<thead>
<tr>
<th></th>
<th>Fresh</th>
<th>Smoked(regular)</th>
<th>Smoked(overtime)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hams</td>
<td>$8</td>
<td>$14</td>
<td>$11</td>
</tr>
<tr>
<td>Bellies</td>
<td>$4</td>
<td>$12</td>
<td>$7</td>
</tr>
<tr>
<td>Picnics</td>
<td>$4</td>
<td>$13</td>
<td>$9</td>
</tr>
</tbody>
</table>

Find the production schedule that maximizes the total net profit, and give a proof that it is the optimal one.

2. Give a pivot rule for the Simplex algorithm which can lead to cycling and an example showing this.

3. Gordon's lemma says that for any $n \times n$ matrix $A$, exactly one of the following holds:
   - $Ax = 0$ for some $x \neq 0$, $x \geq 0$.
   - $y^T A < 0$ for some $y$.

   Here $x$ and $y$ are vectors in $\mathbf{R}^n$. Prove the lemma and give a geometric interpretation.

4. Let $K$ be a convex body (a closed, bounded, convex set) and $x$ be a point. Show that there is a unique point $y \in K$ which minimizes the Euclidean distance from $x$ to some point in $K$ (assume that there exists such a point).

5. Volume computation.
   
   (a) (Bonus) Let $S$ be a convex body in $\mathbf{R}^n$. Let $A$ be an $n \times n$ nonsingular matrix (i.e. $\text{det}(A) \neq 0$). Show that the set $\{Ax : x \in S\}$ is also convex and that its volume is $|\text{det}(A)| \cdot \text{vol}(S)$. 

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(b) An ellipsoid in $\mathbb{R}^n$ is the set of points $\{x|(Ax)^T(Ax) \leq r^2\}$ for some nonsingular $n \times n$ matrix $A$. Show that the volume of an ellipsoid is $\text{vol}(B_{n,r})/|\text{det}(A)|$, where $B_{n,r}$ is the $n$-dimensional ball of radius $r$.

6. Given a complete graph $G = (V,E)$ with a positive length $w_{ij}$ between each pair of vertices $i, j \in V$, the traveling salesman problem is to find a minimum length Hamilton cycle of $G$.

(a) Give an integer linear program to solve the traveling salesman problem.

(b) Relax the integrality constraints to obtain a linear programming relaxation. Show that the linear program can be solved in polynomial-time by designing a separation oracle.