Problem 1. In NMR quantum computing, a Hadamard gate is implemented by rotating around the axis \((\vec{x} + \vec{z})/\sqrt{2}\). Compute the matrix obtained by rotation around this axis by \(\pi\) radians, and compare to a Hadamard gate.

Solution:

If we denote the rotation by angle \(\theta\) about \((\vec{x} + \vec{z})/\sqrt{2}\) by \(R(\theta)\), we have

\[
R(\theta) = \exp[-i(\theta/2)(\sigma_X + \sigma_Z)/\sqrt{2}]
= \cos\theta/2 I - i\sin\theta/2(\sigma_X + \sigma_Z)/\sqrt{2}
\]

\[\Rightarrow \quad R(\pi) = -i(\sigma_X + \sigma_Z)/\sqrt{2}
= \frac{-i}{\sqrt{2}} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}
= \frac{-i}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}
= -iH
\]

where \(H\) is the Hadamard gate.

Problem 2. Let

\[
H = \frac{1}{2}(\sigma_X \otimes \sigma_X + \sigma_Y \otimes \sigma_Y + \sigma_Z \otimes \sigma_Z + I \otimes I)
\]

be an operator on two qubits.

a) Find \(H^2\) and write it in a simple form.

b) Using (a), find \(\exp(-i\pi H/4)\) and \(\exp(-i\pi H/2)\).

c) Find the eigenvalues of \(H\).

d) Find a set of orthonormal eigenstates of \(H\).
Solution:

a) We have

\[ H^2 = \frac{1}{2} (\sigma_X \otimes \sigma_X + \sigma_Y \otimes \sigma_Y + \sigma_Z \otimes \sigma_Z + I \otimes I)H. \]

Note that

\[ \frac{1}{2} (\sigma_X \otimes \sigma_X)H = \frac{1}{4} (\sigma_X \otimes \sigma_X)(\sigma_X \otimes \sigma_X + \sigma_Y \otimes \sigma_Y + \sigma_Z \otimes \sigma_Z + I \otimes I) \]

\[ = \frac{1}{4} (\sigma_X \sigma_X \otimes \sigma_X \sigma_X + \sigma_X \sigma_Y \otimes \sigma_Y \sigma_Y + \sigma_X \sigma_Z \otimes \sigma_Z \sigma_Z + \sigma_X \otimes \sigma_X) \]

\[ = \frac{1}{4} (I \otimes I + i\sigma_Z \otimes i\sigma_Z + (i)\sigma_Y \otimes (i)\sigma_Y + \sigma_X \otimes \sigma_X) \]

\[ = \frac{1}{4} (I \otimes I - \sigma_Z \otimes \sigma_Z - \sigma_Y \otimes \sigma_Y + \sigma_X \otimes \sigma_X). \]

Similarly,

\[ \frac{1}{2} (\sigma_Y \otimes \sigma_Y)H = \frac{1}{4} (-\sigma_X \otimes \sigma_X + \sigma_Y \otimes \sigma_Y - \sigma_Z \otimes \sigma_Z + I \otimes I) \]

\[ \frac{1}{2} (\sigma_Z \otimes \sigma_Z)H = \frac{1}{4} (-\sigma_X \otimes \sigma_X - \sigma_Y \otimes \sigma_Y + \sigma_Z \otimes \sigma_Z + I \otimes I) \]

\[ \frac{1}{2} (I \otimes I)H = \frac{H}{2}. \]

Adding up these four relations, one can obtain

\[ H^2 = I \otimes I. \]

b) Using equation (4.7) of N&C, we have

\[ \exp(i\theta H) = \cos(\theta)I \otimes I + i\sin(\theta)H \]

\[ \Rightarrow \]

\[ \exp(-i\pi H / 4) = \sqrt{2}I \otimes I / 2 - i\sqrt{2}H / 2 \]

and

\[ \exp(-i\pi H / 2) = -iH. \]

c) Using Problem 1(b) in Problem Set 2, it can be seen that the only possible values for the eigenvalues are +1 and –1.

d) You can easily verify that the Bell states, described in the first problem of Problem Set 3, are one possible set of eigenstates. (In fact, \( H = I_{AB}^2 - I \otimes I \).) The first state in that problem, the singlet state, has eigenvalue –1 and the other three have eigenvalues +1.

Problem 3. Let \( N \) be an integer larger than 5. Consider the following state:
\[ |\psi\rangle = \frac{1}{\sqrt{N}} \sum_{x=0}^{N-1} |x \text{ mod } N \rangle_A \otimes |3x \text{ mod } N \rangle_B \otimes |5x \text{ mod } N \rangle_C . \]

Find the output state if we take a quantum Fourier transform modulus \( N \) on each of the registers \( A, B, \) and \( C \). That is, if we denote the corresponding QFT operators to each system by \( U_A, U_B, \) and \( U_C \), find \( U_A \otimes U_B \otimes U_C |\psi\rangle \). Write your answer in the basis \( \{|0\rangle, |1\rangle, \ldots, |N-1\rangle\}^3 \), and show that it is the superposition of equally probable states. What is this probability?

**Solution:**

\[
U_A \otimes U_B \otimes U_C |\psi\rangle = \frac{1}{\sqrt{N}} \sum_{x=0}^{N-1} U_A |x \text{ mod } N \rangle_A \otimes U_B |3x \text{ mod } N \rangle_B \otimes U_C |5x \text{ mod } N \rangle_C \\
= \left( \frac{1}{\sqrt{N}} \right)^3 \sum_{x=0}^{N-1} \sum_{k=0}^{N-1} \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} e^{2\pi i (kA + 3mA + 5mA) / N} |k\rangle_A |m\rangle_B |n\rangle_C \\
= \frac{1}{N^2} \sum_{k=0}^{N-1} \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} e^{2\pi i (k + 3m + 5n) x / N} |k\rangle_A |m\rangle_B |n\rangle_C \\
= \frac{1}{N^2} \sum_{k=0}^{N-1} \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} |k\rangle_A |m\rangle_B |n\rangle_C \sum_{x=0}^{N-1} e^{2\pi i (k + 3m + 5n) x / N} \\
= \frac{1}{N^2} \sum_{k=0}^{N-1} \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} |k\rangle_A |m\rangle_B |n\rangle_C \sum_{x=0}^{N-1} e^{2\pi i (k + 3m + 5n) x / N} \\
= \frac{1}{N} \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} |k\rangle_A |m\rangle_B |n\rangle_C \sum_{x=0}^{N-1} e^{-3m - 5n \text{ mod } N} \\
= \frac{1}{N} \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} |k\rangle_A |m\rangle_B |n\rangle_C .
\]

This is the superposition of \( N^2 \) states each with probability of occurrence \( 1 / N^2 \).