A cluster state is a highly entangled rectangular array of qubits. We measure qubits one at a time. The wiring diagram tells us which basis to measure each qubit in, and which order to measure them in. A wiring diagram is represented by connecting up the dots (which represent qubits) with lines. A junction of two separate lines represents a gate. These gates do not have to be unitary, but if done right, are.

Figure 1: Unconnected cluster states

Figure 2: Wiring diagram for two gates

Measurement on a wiring diagram is done by first measuring all the qubits that are not in the wiring diagram (i.e. unconnected) in the $\sigma_z$ basis. Once those qubits are measured, we measure the qubits in the circuit from left to right in the specified basis.
Cluster state given by eigenvalue equations. The neighborhood of a qubit are the up, down, left, and right qubits.

\[ K^{(a)} = \sigma_x^{(a)} \otimes \bigotimes_b \sigma_z^{(b)} | b \in \text{neighborhood}(a) \]  

The claim is that \( K^{(a)} \) commutes with \( K^{(b)} \) when \( a \neq b \). To show that this is true, we can look at the following cases:

\[ \text{neighborhood}(a) \cap \text{neighborhood}(b) = \emptyset \]  

When this is true, then there is absolutely no overlap between \( K^{(a)} \) and \( K^{(b)} \) and thus the two commute.

\[ \text{neighborhood}(a) \not\ni b \]  

This means that neighborhoods overlap, but that the qubit \( b \) is not in the neighborhood of \( a \) in an arrangement such as

\[
\begin{array}{c}
\cdot \quad a_u \quad \cdot \\
\cdot \quad a_l \quad a \quad a_r \quad b \\
\cdot \quad a_d \quad \cdot
\end{array}
\]

Figure 3: Neighborhoods overlap

\[
K^{(a)} = \sigma_x^{(a)} \otimes \sigma_z^{(a_r)} \otimes \cdots
\]

\[
K^{(b)} = \sigma_x^{(b)} \otimes \sigma_z^{(a_r)} \otimes \cdots
\]

\[
K^{(a)} K^{(b)} = \sigma_x^{(a)} \otimes \sigma_z^{(a_r)} \otimes \cdots \otimes \sigma_x^{(b)} \otimes \sigma_z^{(a_r)} \otimes \cdots
\]

And in the third case, \( a \) and \( b \) are adjacent to each other.

\[
\begin{array}{c}
\cdot \quad a_u \quad \cdot \\
\cdot \quad a_l \quad a \quad b_r \\
\cdot \quad a_d \quad \cdot
\end{array}
\]

Figure 4: \( a \) and \( b \) are adjacent
\[ K^{(a)}K^{(b)} = \sigma_x^{(a)} \otimes \sigma_z^{(b)} \otimes \cdots \otimes \sigma_x^{(a)} \sigma_z^{(b)} \cdots \] (7)

In all three cases \( K^{(a)} \) and \( K^{(b)} \) both commute, so the claim holds. This means that \( K^{(a)} \) are all simultaneously diagonalizable. Any simultaneous eigenvector of \( K^{(a)} \), \( a \in C \) (cluster) is a cluster state. Each \( K^{(a)} \) has eigenvalue \( \pm 1 \), making for \( 2^n \) vectors of eigenvalues \( \{K_a\} \).

\[ |\phi_{\{\kappa_a\}}\rangle_C \] is a cluster state with eigenvalue \( \kappa_a \) on qubit \( a \), \( \{\kappa_a\} = \{\pm 1\} \). Thus \( \langle \phi_{\{\kappa_a\}} | \phi_{\{\kappa'_a\}} \rangle_C = 0 \) if \( \{\kappa_a\} \neq \{\kappa'_a\} \). For example,

\[ \kappa_b = +1 \] (8)
\[ \kappa_a = -1 \] (9)
\[ \langle \phi_{\{\kappa_a\}} | \phi_{\{\kappa'_a\}} \rangle_C = -\langle \phi_{\{\kappa_a\}} | K_b | \phi_{\{\kappa'_a\}} \rangle_C = -\langle \phi_{\{\kappa_a\}} | \phi_{\{\kappa'_a\}} \rangle_C = 0 \] (10)

If \( \{\kappa_a\} = \{\kappa'_a\} \) except for \( \kappa_b = -\kappa'_b \), then \( \sigma_z^{(b)} | \phi_{\{\kappa_a\}} \rangle_C = | \phi_{\{\kappa'_a\}} \rangle_C \)

\[ K_a \sigma_z^{(b)} | \phi_{\{\kappa_a\}} \rangle_C = ( -1 )^{\delta_{ab}} \sigma_z^{(b)} K_a | \phi_{\{\kappa_a\}} \rangle_C \]
\[ = ( -1 )^{\delta_{ab}} \sigma_z^{(b)} K_a | \phi_{\{\kappa_a\}} \rangle_C \]
\[ = ( -1 )^{\delta_{ab}} \sigma_z^{(b)} | \phi_{\{\kappa_a\}} \rangle_C \]

Cluster state for \( \forall_a \kappa_a = 1 \), start in state \( |\psi\rangle_C = \bigotimes_a |+\rangle \), where \( |+\rangle = \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle) \). We then apply \( S_{ab} \) to all neighbors \( a, b \).

\[ S_{ab} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix} \]
\[ = \frac{1}{2} (I + \sigma_x^{(a)} + \sigma_z^{(b)} - \sigma_x^{(a)} \otimes \sigma_z^{(b)}) \] (15)

Here are a few examples of gates that can be made using wiring diagrams:

- **CNOT Gate**: \( \sigma_x \downarrow \sigma_x \sigma_y \downarrow \sigma_y \)
- **Transmission Line**: \( \sigma_x \sigma_x \sigma_x \sigma_x \)
- **Hadamard**: \( \sigma_x \sigma_y \sigma_y \sigma_x \)
- **Rotation**: \( \sigma_x \sigma_x \pm \theta \sigma_y \sigma_x \)

Figure 5: CNOT Gate, Transmission Line, Hadamard, Rotation
We also know that $S_{ab}$ commutes with $S_{a'b'}$. In the $|0\rangle, |1\rangle$ basis, $S_{ab}$ can be represented by a diagonal matrix, which means that they have to commute. $K^{(a)} \otimes_{a,b} S_{ab} |+\rangle^{\otimes n}$ is an eigenvector of $K^{(a)}$.

Demonstration of a transmission line effect:

\[
S_{ab} |+\rangle |+\rangle = S_{ab} \frac{1}{2} (|00\rangle + |01\rangle + |10\rangle + |11\rangle) = \frac{1}{2} (|00\rangle + |01\rangle + |10\rangle - |11\rangle) = \frac{1}{\sqrt{2}} (|+\rangle |0\rangle + |\rangle |1\rangle) \quad (16)
\]

\[
S_{ab} |+\rangle |+\rangle = \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle) \quad (17)
\]

With this, we apply $S_{ab}$ and measure both $a$ and $b$ in the $|+, \rangle$ basis. This is equivalent to measuring in the $\langle ++| S_{ab}, \langle +-| S_{ab}, \langle -+| S_{ab}, \langle --| S_{ab}$ basis, which is also equivalent to measuring in the $\frac{1}{\sqrt{2}} (\langle 0+\rangle + \langle 1-\rangle)$ basis. In this way we get the teleportation effect on the original $|\psi\rangle$. 

Figure 6: Transmission line