18.440 PROBLEM SET SIX DUE APRIL 4

A. FROM TEXTBOOK CHAPTER FIVE:

1. Problem 23: One thousand independent rolls of a fair die will be made. Compute an approximation to the probability that the number 6 will appear between 150 and 200 times inclusively. If the number 6 appears exactly 200 times, find the probability that the number 5 will appear less than 150 times.

2. Problem 27: In 10,000 independent tosses of a coin, the coin lands on heads 5800 times. Is it reasonable to assume that the coin is not fair? Explain.

3. Problem 32: The time (in hours) required to repair a machine is an exponentially distributed random variable with parameter \( \lambda = 1/2 \). What is
   (a) the probability that a repair time exceeds 2 hours?
   (b) the conditional probability that a repair takes at least 10 hours, given that its duration exceeds 9 hours?

4. Theoretical Exercise 9: If \( X \) is an exponential random variable with parameter \( \lambda \), and \( c > 0 \), show that \( cX \) is exponential with parameter \( \lambda/c \).

5. Theoretical Exercise 21: Show that \( \Gamma(1/2) = \sqrt{\pi} \). Hint:
   \[ \Gamma(1/2) = \int_0^\infty e^{-x}x^{-1/2}dx \]
   Make the change of variables \( y = \sqrt{2x} \) and then relate the resulting expression to the normal distribution.

6. Theoretical Exercise 29: Let \( X \) be a continuous random variable having cumulative distribution function \( F \). Define the random variable \( Y \) by \( Y = F(X) \). Show that \( Y \) is uniformly distributed over \((0, 1)\).

7. Theoretical Exercise 30: Let \( X \) have probability density \( f_X \). Find the probability density function of the random variable \( Y \) defined by \( Y = aX + b \).

B. At time zero, a single bacterium in a dish divides into two bacteria. This species of bacteria has the following property: after a bacterium \( B \) divides into two new bacteria \( B_1 \) and \( B_2 \), the subsequent length of time until each \( B_i \) divides is an exponential random variable of rate \( \lambda = 1 \), independently of everything else happening in the dish.
1. Compute the expectation of the time $T_n$ at which the number of bacteria reaches $n$.

2. Compute the variance of $T_n$.

3. Are both of the answers above unbounded, as functions of $n$? Give a rough numerical estimate of the values when $n = 10^{50}$.