18.440 Final Exam: 100 points
Carefully and clearly show your work on each problem (without writing anything that is technically not true) and put a box around each of your final computations.

1. (10 points) Let $X$ be the number on a standard die roll (i.e., each of \{1, 2, 3, 4, 5, 6\} is equally likely) and $Y$ the number on an independent standard die roll. Write $Z = X + Y$.

1. Compute the condition probability $P[X = 4|Z = 6]$. \textbf{ANSWER:} 1/5


2. (10 points) Janet is standing outside at time zero when it starts to drizzle. The times at which raindrops hit her are a Poisson point process with parameter $\lambda = 2$. In expectation, she is hit by 2 raindrops in each given second.

(a) What is the expected amount of time until she is first hit by a raindrop? \textbf{ANSWER:} 1/2 second

(b) What is the probability that she is hit by exactly 4 raindrops during the first 2 seconds of time? \textbf{ANSWER:} $e^{-2\lambda}(2\lambda)^k/k! = e^{-4}4^4/4!$.

3. (10 points) Let $X$ be a random variable with density function $f$, cumulative distribution function $F$, variance $V$ and mean $M$.

(a) Compute the mean and variance of $3X + 3$ in terms of $V$ and $M$. \textbf{ANSWER:} Mean $3M + 3$, variance $9V$.

(b) If $X_1, \ldots, X_n$ are independent copies of $X$, Compute (in terms of $F$) the cumulative distribution function for the largest of the $X_i$. \textbf{ANSWER:} $F(a)^n$. This is the probability that all $n$ values are less than $a$.

4. (10 points) Suppose that $X_i$ are i.i.d. random variables, each uniform on [0, 1]. Compute the moment generating function for the sum $\sum_{i=1}^{n} X_i$. \textbf{ANSWER:} $M_{aX_1} = E^{ax} X_1 = \int_0^1 e^{ax} dx = (e^a - 1)/a$. Moment generating function for sum is $(e^a - 1)^n/a^n$.

5. (10 points) Suppose that $X$ and $Y$ are outcomes of independent standard die rolls (each equal to \{1, 2, 3, 4, 5, 6\} with equal probability). Write $Z = X + Y$.
(a) Compute the entropies $H(X)$ and $H(Y)$. **Answer:** $\log 6$ and $\log 6$

(b) Compute $H(X, Z)$. **Answer:** $\log 36 = 2 \log 6$.

(c) Compute $H(10X + Y)$. **Answer:** $\log 36 = 2 \log 6$ (since 36 sums all distinct).

(d) Compute $H(Z) + H_Z(Y)$. (Hint: you shouldn’t need to do any more calculations.) **Answer:** $\log 36 = 2 \log 6$.

6. (10 points) Elaine’s not-so-trusty old car has three states: broken (in Elaine’s possession), working (in Elaine’s possession), and in the shop. Denote these states B, W, and S.

(i) Each morning the car starts out B, it has a .5 chance of staying B and a .5 chance of switching to S by the next morning.

(ii) Each morning the car starts out W, it has .5 chance of staying W, and a .5 chance of switching to B by the next morning.

(iii) Each morning the car starts out S, it has a .5 chance of staying S and a .5 chance of switching to W by the next morning.

Answer the following

(a) Write the three-by-three Markov transition matrix for this problem. **Answer:** Markov chain matrix is

$$M = \begin{pmatrix} .5 & 0 & .5 \\ .5 & .5 & 0 \\ 0 & .5 & .5 \end{pmatrix}$$

(b) If the car starts out B on one morning, what is the probability that it will start out B two days later? **Answer:** 1/4

(c) Over the long term, what fraction of mornings does the car start out in each of the three states, B, S, and W? **Answer:** Row vector $\pi$ such that $\pi M = \pi$ (with components of $\pi$ summing to one) is

$$\left( \frac{1}{3} \quad \frac{1}{3} \quad \frac{1}{3} \right).$$

7. Suppose that $X_1, X_2, X_3, \ldots$ is an infinite sequence of independent random variables which are each equal to 2 with probability $1/3$ and $.5$ with probability $2/3$. Let $Y_0 = 1$ and $Y_n = \prod_{i=1}^{n} X_i$ for $n \geq 1$. 

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(a) What is the probability that $Y_n$ reaches 8 before the first time that it reaches $\frac{1}{8}$? ANSWER: sequences is martingale, so $1 = EY_T = 8p + (1/8)(1 - p)$. Solving gives $1 - 8p = (1 - p)/8$, so $8 - 64p = 1 - p$ and $63p = 7$. Answer is $p = 1/9$.

(b) Find the mean and variance of $\log Y_{10000}$. ANSWER: Compute for $\log Y_1$, multiply by 10000.

(c) Use the central limit theorem to approximate the probability that $\log Y_{10000}$ (and hence $Y_{10000}$) is greater than its median value. ANSWER: About $.5$.

8. (10 points) Eight people toss their hats into a bin and the hats are redistributed, with all of the $8!$ hat permutations being equally likely. Let $N$ be the number of people who get their own hat. Compute the following:

(a) $E[N]$ ANSWER: 1

(b) $\text{Var}[N]$ ANSWER: 1

9. (10 points) Let $X$ be a normal random variable with mean $\mu$ and variance $\sigma^2$.

(a) $E e^X$. ANSWER: $e^{\mu + \sigma^2/2}$.

(b) Find $\mu$, assuming that $\sigma^2 = 3$ and $E[e^X] = 1$. ANSWER: $\mu + \sigma^2/2 = 0$ so $\mu = -9/2$.

10. (10 points)

1. Let $X_1, X_2, \ldots$ be independent random variables, each equal to 1 with probability $1/2$ and $-1$ with probability $1/2$. In which of the cases below is the sequence $Y_n$ a martingale? (Just circle the corresponding letters.)

   (a) $Y_n = X_n$ NO
   (b) $Y_n = 1 + X_n$ NO
   (c) $Y_n = 7$ YES
   (d) $Y_n = \sum_{i=1}^{n} iX_i$ YES
   (e) $Y_n = \prod_{i=1}^{n} (1 + X_i)$ YES

2. Let $Y_n = \sum_{i=1}^{n} X_i$. Which of the following is necessarily a stopping time for $Y_n$?
(a) The smallest $n$ for which $|Y_n| = 5$. **YES**
(b) The largest $n$ for which $Y_n = 12$ and $n < 100$. **NO**
(c) The smallest value $n$ for which $n > 100$ and $Y_n = 12$. **YES**