18.440 Final Exam: 100 points
Carefully and clearly show your work on each problem (without writing anything that is technically not true) and put a box around each of your final computations.
1. (10 points) Let $X$ be the number on a standard die roll (i.e., each of \{1, 2, 3, 4, 5, 6\} is equally likely) and $Y$ the number on an independent standard die roll. Write $Z = X + Y$.


2. Compute the conditional expectation $E[Z | Y]$ as a function of $Y$. 
2. (10 points) Janet is standing outside at time zero when it starts to drizzle. The times at which raindrops hit her are a Poisson point process with parameter $\lambda = 2$. In expectation, she is hit by 2 raindrops in each given second.

(a) What is the expected amount of time until she is first hit by a raindrop?

(b) What is the probability that she is hit by exactly 4 raindrops during the first 2 seconds of time?
3. (10 points) Let $X$ be a random variable with density function $f$, cumulative distribution function $F$, variance $V$ and mean $M$.

(a) Compute the mean and variance of $3X + 3$ in terms of $V$ and $M$.

(b) If $X_1, \ldots, X_n$ are independent copies of $X$. Compute (in terms of $F$) the cumulative distribution function for the largest of the $X_i$. 

4. (10 points) Suppose that $X_i$ are i.i.d. random variables, each uniform on $[0, 1]$. Compute the moment generating function for the sum $\sum_{i=1}^n X_i$. 


5. (10 points) Suppose that $X$ and $Y$ are outcomes of independent standard die rolls (each equal to \{1, 2, 3, 4, 5, 6\} with equal probability). Write $Z = X + Y$.

(a) Compute the entropies $H(X)$ and $H(Y)$.

(b) Compute $H(X, Z)$.

(c) Compute $H(10X + Y)$.

(d) Compute $H(Z) + H_Z(Y)$. (Hint: you shouldn’t need to do any more calculations.)
6. (10 points) Elaine’s not-so-trusty old car has three states: broken (in Elaine’s possession), working (in Elaine’s possession), and in the shop. Denote these states B, W, and S.

(i) Each morning the car starts out B, it has a .5 chance of staying B and a .5 chance of switching to S by the next morning.

(ii) Each morning the car starts out W, it has .5 chance of staying W, and a .5 chance of switching to B by the next morning.

(iii) Each morning the car starts out S, it has .5 chance of staying S and a .5 chance of switching to W by the next morning.

Answer the following

(a) Write the three-by-three Markov transition matrix for this problem.

(b) If the car starts out B on one morning, what is the probability that it will start out B two days later?

(c) Over the long term, what fraction of mornings does the car start out in each of the three states, B, S, and W?
7. Suppose that $X_1, X_2, X_3, \ldots$ is an infinite sequence of independent random variables which are each equal to 2 with probability $1/3$ and 0.5 with probability $2/3$. Let $Y_0 = 1$ and $Y_n = \prod_{i=1}^{n} X_i$ for $n \geq 1$.

(a) What is the probability that $Y_n$ reaches 8 before the first time that it reaches $\frac{1}{8}$?

(b) Find the mean and variance of $\log Y_{10000}$.

(c) Use the central limit theorem to approximate the probability that $\log Y_{10000}$ (and hence $Y_{10000}$) is greater than its median value.
8. (10 points) Eight people toss their hats into a bin and the hats are redistributed, with all of the 8! hat permutations being equally likely. Let \( N \) be the number of people who get their own hat. Compute the following:

(a) \( \mathbb{E}[N] \)

(b) \( \text{Var}[N] \)
9. (10 points) Let $X$ be a normal random variable with mean $\mu$ and variance $\sigma^2$.

(a) $Ee^X$.

(b) Find $\mu$, assuming that $\sigma^2 = 3$ and $E[e^X] = 1$. 
10. (10 points)

1. Let $X_1, X_2, \ldots$ be independent random variables, each equal to 1 with probability $1/2$ and $-1$ with probability $1/2$. In which of the cases below is the sequence $Y_n$ a martingale? (Just circle the corresponding letters.)

(a) $Y_n = X_n$
(b) $Y_n = 1 + X_n$
(c) $Y_n = 7$
(d) $Y_n = \sum_{i=1}^{n} iX_i$
(e) $Y_n = \prod_{i=1}^{n} (1 + X_i)$

2. Let $Y_n = \sum_{i=1}^{n} X_i$. Which of the following is necessarily a stopping time for $Y_n$?

(a) The smallest $n$ for which $|Y_n| = 5$.
(b) The largest $n$ for which $Y_n = 12$ and $n < 100$.
(c) The smallest value $n$ for which $n > 100$ and $Y_n = 12$. 