18.440 Midterm 1, Spring 2011: 50 minutes, 100 points.  
SOLUTIONS

1. (20 points) Consider an infinite sequence of independent tosses of a coin that comes up heads with probability $p$.

(a) Let $X$ be such that the first heads appears on the $X$th toss. In other words, $X$ is the number of tosses required to obtain a heads. Compute (in terms of $p$) the expectation $E[X]$. ANSWER: geometric random variable with parameter $p$ has expectation $1/p$.

(b) Compute (in terms of $p$) the probability that exactly 5 of the first 10 tosses are heads. ANSWER: binomial probability $\binom{10}{5} p^5(1-p)^5$

(c) Compute (in terms of $p$) the probability that the 5th head appears on the 10th toss. ANSWER: negative binomial. Need 4 heads in first 9 tosses, 10th toss heads. Probability $\binom{9}{4} p^4(1-p)^5p$.

2. (20 points) Jill sends her resume to 1000 companies she finds on monster.com. Each company responds with probability 3/1000 (independently of what all the other companies do). Let $R$ be the number of companies that respond.


(b) Compute $\text{Var}[R]$. ANSWER: binomial random variable with $n = 1000$ and $p = 3/1000$. $\text{Var}[R] = np(1-p) = 3(1 - 3/1000)$.

(c) Use a Poisson random variable approximation to estimate the probability $P\{R = 3\}$. ANSWER: $R$ is approximately Poisson with $\lambda = 3$. So $P\{R = 3\} \approx e^{-\lambda}\lambda^3/k! = e^{-3}3^3/3! = 9e^{-3}/2$.

3. (10 points) How many four-tuples $(x_1, x_2, x_3, x_4)$ of non-negative integers satisfy $x_1 + x_2 + x_3 + x_4 = 10$? ANSWER: represent partition with stars and bars $\ast\ast\ast\mid\ast\ast\ast$| $\ast\ast\ast\ast\ast$| $\ast\ast\ast\ast\ast$. Have $\binom{13}{3}$ ways to do this.

4. (10 points) Suppose you buy a lottery ticket that gives you a one in a million chance to win a million dollars. Let $X$ be the amount you win. Compute the following:

(a) $E[X]$. ANSWER: $\frac{1}{10^6}10^6 = 1$.

(b) $\text{Var}[X]$. ANSWER: $E[X^2] - E[X]^2 = \frac{1}{10^6}(10^6)^2 - 1^2 = 10^6 - 1$. 

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5. (20 points) Suppose that $X$ is continuous random variable with probability density function $f_X(x) = \begin{cases} 2x & x \in [0, 1] \\ 0 & x \notin [0, 1] \end{cases}$. Compute the following:

(a) The expectation $E[X]$. ANSWER: $\int_{-\infty}^{\infty} f_X(x) dx = \int_{0}^{1} f_X(x) dx = \int_{0}^{1} 2x^2 dx = \frac{2}{3}x^3 \bigg|_{0}^{1} = \frac{2}{3}$. 

(b) The variance $\text{Var}[X]$. ANSWER: $E[X^2] = \int_{-\infty}^{\infty} f_X(x) x^2 dx = \int_{0}^{1} f_X(x) x^2 dx = \int_{0}^{1} 2x^3 dx = \frac{2}{3}x^4 \bigg|_{0}^{1} = 1/2$. So variance is $1/2 - (2/3)^2 = 1/2 - 4/9 = 1/18$.

(c) The cumulative distribution function $F_X$. ANSWER: $F_X(a) = \int_{-\infty}^{a} f_X(x) dx = \begin{cases} 0 & a < 0 \\ a^2 & a \in [0, 1] \\ 1 & a > 1 \end{cases}$.

6. (20 points) A standard deck of 52 cards contains 4 aces. Suppose we choose a random ordering (all 52! permutations being equally likely). Compute the following:

(a) The probability that all of the top 4 cards in the deck are aces. ANSWER: $4!$ ways to order aces, $48!$ ways to order remainder. Probability $4!48!/52!$.

(b) The probability that none of the top 4 cards in the deck is an ace. ANSWER: choose cards one at a time starting at the top and multiply number of available choices at each stage to get total number. Probability is $48 \cdot 47 \cdot 46 \cdot 45 \cdot 48!/52!$.

(c) The expected number of aces among the top 4 cards in the deck. (There is a simple form for the solution.) ANSWER: have probability $4/52 = 1/13$ that top card is an ace. Similarly, probability $1/13$ that $j$th card is an ace for each $j \in \{1, 2, 3, 4\}$. Additivity of expectation gives answer: $4/13$. 
