1. (10 points) Suppose that a fair die is rolled 18000 times. Each roll turns up a uniformly random member of the set \{1, 2, 3, 4, 5, 6\} and the rolls are independent of each other. Let \(X\) be the total number of times the die comes up 1.

(a) Compute \(\text{Var}(X)\). \textbf{ANSWER:} \(npq = 18000(5/6)(1/6) = 2500\)

(b) Use a normal random variable approximation to estimate the probability that \(X < 2900\). You may use the function \(\Phi(a) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a} e^{-x^2/2}dx\) in your answer. \textbf{ANSWER:} Standard derivation is \(\sqrt{2500} = 50\). Probability \(X\) more than 2 standard deviations below mean is approximately \(\Phi(-2)\).

2. (20 points) Let \(X_1, X_2,\) and \(X_3\) be independent uniform random variables on \([0, 1]\). Write \(Y = X_1 + X_2\) and \(Z = X_2 + X_3\).

(a) Compute \(\mathbb{E}[X_1X_2X_3]\). \textbf{ANSWER:} Independence implies \(\mathbb{E}[X_1X_2X_3] = \mathbb{E}[X_1]\mathbb{E}[X_2]\mathbb{E}[X_3] = (1/2)^3 = 1/8\).

(b) Compute \(\text{Var}(X_1)\). \textbf{ANSWER:} \(\mathbb{E}[X_1^2] = \int_0^1 x^2 dx = 1/3\), so \(\text{Var}(X_1) = \mathbb{E}[X_1^2] - (\mathbb{E}[X_1])^2 = 1/3 - 1/4 = 1/12\).

(c) Compute the covariance \(\text{Cov}(Y, Z)\) and the correlation coefficient \(\rho(Y, Z)\). \textbf{ANSWER:} By bilinearity of covariance,

\[
\text{Cov}(Y, Z) = \text{Cov}(X_1 + X_2, X_2 + X_3) = \text{Cov}(X_1, X_2) + \text{Cov}(X_1, X_3) + \text{Cov}(X_2, X_2) + \text{Cov}(X_2, X_3).
\]

All terms are zero by independence except \(\text{Cov}(X_2, X_2) = \text{Var}(X_2) = 1/12\). Then \(\rho(Y, Z) = \frac{1/12}{\sqrt{1/2}(2/12)} = 1/2\).

(d) Compute and draw a graph of the density function \(f_Y\). \textbf{ANSWER:} \(f_Y(a) = \int_{-\infty}^{\infty} f_X(a - y)f_X(y)dy\) where

\[
f_X(x) = \begin{cases} 1 & x \in (0, 1) \\ 0 & x \notin (0, 1) \end{cases}.
\]

Then

\[
f_X(a - y)f_X(y) = \begin{cases} 1 & a - y \in (0, 1), y \in (0, 1) \\ 0 & \text{otherwise} \end{cases}.
\]
Now \( a - y \in (0, 1) \) is equivalent to \(-y \in (-a, 1 - a)\) or equivalently \( y \in (a - 1, a)\). Thus \( f_Y(a) \) is equal to the length of the intersection of the intervals \((0, 1)\) and \((a - 1, a)\). This becomes

\[
f_Y(a) = \begin{cases} 
0 & a < 0 \\
\frac{a}{2} & 0 \leq a < 1 \\
2 - a & 1 \leq a < 2 \\
0 & a \geq 2
\end{cases}
\]

3. (20 points) Suppose that \( X_1, X_2, \ldots, X_n \) are independent uniform random variables on \([0, 1]\).

(a) Write \( Y = \min\{X_1, X_2, \ldots, X_n\}\). Compute the cumulative distribution function \( F_Y(a) \) and the density function \( f_Y(a) \) for \( a \in [0, 1] \).

**ANSWER:** By independence, 
\[ P(\min\{X_1, X_2, \ldots, X_n\} > a) = P(X_1 > a)P(X_2 > a)\cdots P(X_n > a) = (1 - a)^n \]

So 
\[ F_Y(a) = 1 - (1 - a)^n, \text{ and } f_Y(a) = F_Y'(a) = n(1 - a)^{n-1} \]

(b) Compute \( P(X_1 < .3) \) and \( P(\max\{X_1, X_2, \ldots, X_n\}) < .3 \).

**ANSWER:** .3 and \(.3^n\).

(c) Compute the expectation \( E[X_1 + X_2 + \ldots + X_n] \).

**ANSWER:** By additivity of expectation, this is \( nE[X_1] = n/2 \).

4. (20 points) Aspiring writer Rachel decides to lock herself in her room to think of screenplay ideas. When Rachel is thinking, the moments at which good new ideas occur to her form a Poisson process with parameter \( \lambda_G = .5/\text{hour} \). The times when bad new ideas occur to her are a Poisson point process with parameter \( \lambda_B = 1.5/\text{hour} \).

(a) Let \( T \) be the amount of time until Rachel has her first idea (good or bad). Write down the probability density function for \( T \).
**ANSWER:** $T$ is exponential with parameter $\lambda = \lambda_G + \lambda_B = 2$, so $f_T(x) = 2e^{-2x}$.

(b) Compute the probability that Rachel has exactly 3 bad ideas total during her first hour of thinking. **ANSWER:** Number $N$ of bad ideas is Poisson with rate $1 \cdot \lambda_B = 1.5$. So $P(N = 3) = \frac{(1.5)^3 e^{-1.5}}{3!}$.

(c) Let $S$ be the amount of time elapsed before the fifth good idea occurs. Compute $\text{Var}(S)$. **ANSWER:** Variance of time till one good idea is $1/\lambda_G^2$. Memoryless property and additivity of variance of independent sums gives $\text{Var}(S) = 5/\lambda_G^2 = 20$.

(d) What is the probability that Rachel has no ideas at all during her first three hours of thinking? **ANSWER:** Time till first idea is exponential with $\lambda = 2$. Probability this time exceeds 3 is $e^{-2 \cdot 3} = e^{-6}$.

5. (20 points) Suppose that $X$ and $Y$ have a joint density function $f$ given by

$$ f(x, y) = \begin{cases} 1/\pi & x^2 + y^2 < 1 \\ 0 & x^2 + y^2 \geq 1 \end{cases}. $$

(a) Compute the probability density function $f_X$ for $X$. **ANSWER:**

$$ f_X(x) = \int_{-\infty}^{\infty} f(x, y) dy = \begin{cases} \frac{1}{\pi} 2\sqrt{1-x^2} & -1 \leq x \leq -1 \\ 0 & \text{otherwise} \end{cases} $$

(b) Compute the conditional expectation $E[X|Y = .5]$. **ANSWER:** Probability density for $X$ given $Y = .5$ is uniform on $(-\sqrt{1-.5^2}, \sqrt{1-.5^2})$. So $E[X|Y = .5] = 0$.

(c) Express $E[X^3Y^3]$ as a double integral. (You don’t have to explicitly evaluate the integral.) **ANSWER:**

$$ \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{\pi} x^3 y^3 dy dx. $$

6. (10 points) Let $X$ and $Y$ be independent normal random variables, each with mean 1 and variance 9.
(a) Let $f$ be the joint probability density function for the pair $(X, Y)$. Write an explicit formula for $f$. **ANSWER:**

$$f(x, y) = \frac{1}{3\sqrt{2\pi}} e^{-(x-1)^2/18} \frac{1}{3\sqrt{2\pi}} e^{-(y-1)^2/18} = \frac{1}{18\pi} e^{-(x-1)^2/(18\pi) - (y-1)^2/(18\pi)}.$$ 

(b) Compute $E[X^2]$ and $E[X^2Y^2]$. **ANSWER:**

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