18.440 Practice Midterm: 50 minutes, 100 points
Carefully and clearly show your work on each problem (without writing anything that is technically not true) and put a box around each of your final computations.

1. (30 points) Twenty people in a room each have independently random birthdays among 365 possibilities. Let $P$ be the number of pairs of people that share a birthday (i.e., the number of ways of choosing a pair of two people that share a birthday). Let $T$ be the number ways of choosing a triple of three people that share a birthday. (If everyone has the same birthday, then $P = 20 \times 19/2$ and $T = 20 \times 19 \times 18/6$..) Compute the following:

(a) Write $E_{i,j}$ for event that $i$th and $j$th person have same birthday. Then $P = \sum_{i<j} E_{i,j}$ and $E[P] = \sum_{i<j} E[1_{E_{i,j}}] = \binom{20}{2} \frac{1}{365}$.

(b) $\text{Var}[P] = E[P^2] - (E[P])^2$ so suffices to compute $E[P^2]$. We have

$$E[P^2] = E[\sum_{i<j} 1_{E_{i,j}} \sum_{k<\ell} 1_{E_{k,\ell}}] = \sum_{i<j} \sum_{k<\ell} E[1_{E_{i,j}} 1_{E_{k,\ell}}].$$

The terms $E[1_{E_{i,j}} 1_{E_{k,\ell}}]$ are $\frac{1}{365}$ if $(i, j) = (k, \ell)$ and $\frac{1}{365}^2$ otherwise. There are $\binom{20}{2}$ terms of former type and $\binom{20}{2}^2 - \binom{20}{2}$ of latter, so $E[P^2] = \binom{20}{2} \frac{1}{365} + \left(\binom{20}{2}^2 - \binom{20}{2}\right) \frac{1}{365^2}$.

(c) Similar arguments to case (a) give $E[T] = \binom{20}{3} \frac{1}{365^3}$.

(d) We can only have $P = 5$ and $T = 1$ if we have one triple, two pairs, and 13 singletons. We count ways to do this in stages: $\binom{20}{3}$ ways to choose people to belong to triple. Then $\binom{17}{2}$ ways to choose people for first pair then $\binom{15}{2}$ ways to choose people for second pair. Given these choices, have 365!/349! ways to assign birthdays to each of the 16 sets. And we overcounted by a factor of 2 (since our designation of “first pair” and “second pair” is arbitrary). So the probability is

$$\frac{\binom{20}{3} \binom{17}{2} \binom{15}{2} (365!/349!)/2}{365^{20}}.$$ 

(e) We need five pairs and 10 singletons. We have $\binom{20}{2} \binom{18}{2} \binom{16}{2} \binom{14}{2} \binom{12}{2} / 5!$ ways to designate the pairs (dividing by 5! since ordering of pairs is
arbitrary). Given these choices, have \(365!/350!\) ways to assign birthdays to each of the 15 sets. So the probability is

\[
\frac{(365!/350!)(\binom{20}{2})^{18}(\binom{16}{2})^{14}(\binom{12}{2})/5!}{365^{20}}.
\]

(f) The probability that \(P = 5\) and \(T \geq 1\) is the same as the probability that \(P = 5\) and \(T = 1\) (computed in (d)) since we cannot have \(P = 5\) if \(T \geq 2\).

2. (20 points) Compute how many:

(a) Quadruples \((w, x, y, z)\) of non-negative integers with \(w + x + y + z = 50\). There are \(\binom{53}{50}\) of these.

(b) Ways to divide 15 books into five groups of size 1, 2, 3, 4, and 5. There are \(\binom{15}{1,2,3,4,5}\) of these.

(c) “Two pair” poker hands: (i.e. 2 cards of one denomination, 2 of another distinct denomination, and one of a third distinct denomination).

3. (20 points)

(a) Roll three dice. Find the probability that there are at least two sixes given that there is at least one six. Probability of zero sixes is \(p_0 = (5/6)^3\). Probability of one six is \(p_1 = 3(5/6)^2(1/6)\). Probability of two or more sixes is \(1 - p_0 - p_1\). Answer is \((1 - p_0 - p_1)/(1 - p_0)\).

(b) Find the conditional probability that a standard poker hand has at least 3 aces given that it has at least 2. Just compute explicitly the probabilities \(p_2, p_3, p_4\) of having 2, 3, 4 aces. Answer is \((p_3 + p_4)/(p_2 + p_3 + p_4)\).

4. (10 points) Suppose that the sample space \(S\) contains three elements \(\{1, 2, 3\}\), with probabilities .5, .2, and .3 respectively. Suppose \(X(s) = s^2 - 4\) for \(s \in S\). Compute

(a) \(E(X)\). Straightforward arithmetic.

(b) \(\text{Cov}(X, |X|)\). Straightforward arithmetic.
5. (20 points) Suppose $X$ is Poissonian random variable with parameter $\lambda_1 = 1$, $Y$ is an independent Poissonian random variable with $\lambda_2 = 2$, and $Z$ is a Poissonian random variable with parameter $\lambda_3 = 3$. Assume $X$ and $Y$ and $Z$ are independent and compute the following:

(a) Trick is to note that $X + Y + Z$ is also Poisson with parameter $\lambda = \lambda_1 + \lambda_2 + \lambda_3 = 6$. So $P\{X + Y + Z = 8\} = e^{-\lambda} \frac{\lambda^8}{8!} = e^{-6} \frac{6^8}{8!}$.

(b) $\text{Cov}(X + 2Y, 2Y + 3Z)$ Trick is to use the bilinearity of covariance and fact that independent variables have zero covariance. We get $\text{Cov}(X + 2Y, 2Y + 3Z) = \text{Cov}(2Y, 2Y) = \text{Var}(2Y) = 4\text{Var}(Y) = 4\lambda_2 = 8$.

(c) $E[XYZ]$ is (since independence implies expectation of product is product of expectations) given by $\lambda_1 \lambda_2 \lambda_3 = 6$.

(d) $E[X^2Y^2Z]$ is (by same reasoning and recalling formula for second moment of Poisson random variable) given by $(\lambda_1^2 + \lambda_1)(\lambda_2^2 + \lambda_2)\lambda_3$. 