Outline

Geometric random variables

Negative binomial random variables

Problems
Outline

Geometric random variables

Negative binomial random variables

Problems
Consider an infinite sequence of independent tosses of a coin that comes up heads with probability $p$.

Let $X$ be such that the first heads is on the $X$th toss.

For example, if the coin sequence is $T, T, H, T, H, T, \ldots$ then $X = 3$.

Then $X$ is a random variable. What is $P\{X = k\}$?

Answer: $P\{X = k\} = (1 - p)^{k-1}p = q^{k-1}p$, where $q = 1 - p$ is tails probability.

Can you prove directly that these probabilities sum to one?

Say $X$ is a geometric random variable with parameter $p$. 

18.440 Lecture 14
Let $X$ be a geometric with parameter $p$, i.e., $P\{X = k\} = (1 - p)^{k-1}p = q^{k-1}p$ for $k \geq 1$.

What is $E[X]$?

By definition $E[X] = \sum_{k=1}^{\infty} q^{k-1}pk$.

There’s a trick to computing sums like this.

Note $E[X - 1] = \sum_{k=1}^{\infty} q^{k-1}p(k - 1)$. Setting $j = k - 1$, we have $E[X - 1] = q \sum_{j=0}^{\infty} q^{j-1}pj = qE[X]$.

Kind of makes sense. $X - 1$ is “number of extra tosses after first.” Given first coin heads (probability $p$), $X - 1$ is 0. Given first coin tails (probability $q$), conditional law of $X - 1$ is geometric with parameter $p$. In latter case, conditional expectation of $X - 1$ is same as a priori expectation of $X$.

Thus $E[X] - 1 = E[X - 1] = p \cdot 0 + qE[X] = qE[X]$ and solving for $E[X]$ gives $E[X] = 1/(1 - q) = 1/p$. 

18.440 Lecture 14
Let $X$ be a geometric random variable with parameter $p$. Then $P\{X = k\} = q^{k-1}p$.

What is $E[X^2]$?

By definition $E[X^2] = \sum_{k=1}^{\infty} q^{k-1} pk^2$.

Let’s try to come up with a similar trick.

Note $E[(X - 1)^2] = \sum_{k=1}^{\infty} q^{k-1} p(k - 1)^2$. Setting $j = k - 1$, we have $E[(X - 1)^2] = q \sum_{j=0}^{\infty} q^{j-1} p j^2 = qE[X^2]$.


$Var[X] = (2-p)/p^2 - 1/p^2 = (1-p)/p^2 = 1/p^2 - 1/p = q/p^2$. 

Geometric random variable variance
Example

- Toss die repeatedly. Say we get 6 for first time on $X$th toss.
- What is $P\{X = k\}$?
  - Answer: $(5/6)^{k-1}(1/6)$.
- What is $E[X]$?
  - Answer: 6.
- What is $\text{Var}[X]$?
  - Answer: $1/p^2 - 1/p = 36 - 6 = 30$.
- Takes $1/p$ coin tosses on average to see a heads.
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Problems
Consider an infinite sequence of independent tosses of a coin that comes up heads with probability $p$.

Let $X$ be such that the $r$th heads is on the $X$th toss.

For example, if $r = 3$ and the coin sequence is $T, T, H, H, T, T, H, T, T, \ldots$ then $X = 7$.

Then $X$ is a random variable. What is $P\{X = k\}$?

Answer: need exactly $r - 1$ heads among first $k - 1$ tosses and a heads on the $k$th toss.

So $P\{X = k\} = \binom{k-1}{r-1} p^{r-1}(1 - p)^{k-r} p$. Can you prove these sum to 1?

Call $X$ **negative binomial random variable with parameters** $(r, p)$. 

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18.440 Lecture 14
Consider an infinite sequence of independent tosses of a coin that comes up heads with probability $p$.

Let $X$ be such that the $r$th heads is on the $X$th toss.

Then $X$ is a **negative binomial random variable with parameters** $(r, p)$.

What is $E[X]$?

Write $X = X_1 + X_2 + \ldots + X_r$ where $X_k$ is number of tosses (following $(k - 1)$th head) required to get $k$th head. Each $X_k$ is geometric with parameter $p$.

So $E[X] = E[X_1 + X_2 + \ldots + X_r] = E[X_1] + E[X_2] + \ldots + E[X_r] = r/p$.

How about $\text{Var}[X]$?

Turns out that $\text{Var}[X] = \text{Var}[X_1] + \text{Var}[X_2] + \ldots + \text{Var}[X_r]$. So $\text{Var}[X] = rq/p^2$. 
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Problems
Nate and Natasha have beautiful new baby. Each minute with .01 probability (independent of all else) baby cries.

**Additivity of expectation:** How many times do they expect the baby to cry between 9 p.m. and 6 a.m.?

**Geometric random variables:** What’s the probability baby is quiet from midnight to three, then cries at exactly three?

**Geometric random variables:** What’s the probability baby is quiet from midnight to three?

**Negative binomial:** Probability fifth cry is at midnight?

**Negative binomial expectation:** How many minutes do I expect to wait until the fifth cry?

**Poisson approximation:** Approximate the probability there are exactly five cries during the night.

**Exponential random variable approximation:** Approximate probability baby quiet all night.
More fun problems

- Suppose two soccer teams play each other. One team’s number of points is Poisson with parameter $\lambda_1$ and other’s is independently Poisson with parameter $\lambda_2$. (You can google “soccer” and “Poisson” to see the academic literature on the use of Poisson random variables to model soccer scores.) Using Mathematica (or similar software) compute the probability that the first team wins if $\lambda_1 = 2$ and $\lambda_2 = 1$. What if $\lambda_1 = 2$ and $\lambda_2 = .5$?

- Imagine you start with the number 60. Then you toss a fair coin to decide whether to add 5 to your number or subtract 5 from it. Repeat this process with independent coin tosses until the number reaches 100 or 0. What is the expected number of tosses needed until this occurs?