18.440: Lecture 15

Continuous random variables

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Outline

Continuous random variables

Expectation and variance of continuous random variables

Measurable sets and a famous paradox
Continuous random variables

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Measurable sets and a famous paradox
Continuous random variables

Say $X$ is a **continuous random variable** if there exists a **probability density function** $f = f_X$ on $\mathbb{R}$ such that $P\{X \in B\} = \int_B f(x)dx := \int 1_B(x)f(x)dx$.

We may assume $\int_{\mathbb{R}} f(x)dx = \int_{-\infty}^{\infty} f(x)dx = 1$ and $f$ is non-negative.

Probability of interval $[a, b]$ is given by $\int_a^b f(x)dx$, the area under $f$ between $a$ and $b$.

Probability of any single point is zero.

Define **cumulative distribution function** $F(a) = F_X(a) := P\{X < a\} = P\{X \leq a\} = \int_{-\infty}^{a} f(x)dx$.  

18.440 Lecture 15
Simple example

Suppose $f(x) = \begin{cases} 1/2 & x \in [0, 2] \\ 0 & x \notin [0, 2]. \end{cases}$

What is $P\{X < 3/2\}$?

What is $P\{X = 3/2\}$?

What is $P\{1/2 < X < 3/2\}$?

What is $P\{X \in (0, 1) \cup (3/2, 5)\}$?

What is $F$?

We say that $X$ is uniformly distributed on the interval $[0, 2]$. 

18.440 Lecture 15
Another example

Suppose \( f(x) = \begin{cases} \frac{x}{2} & x \in [0, 2] \\ 0 & 0 \notin [0, 2]. \end{cases} \)

What is \( P\{X < \frac{3}{2}\} \)?

What is \( P\{X = \frac{3}{2}\} \)?

What is \( P\{\frac{1}{2} < X < \frac{3}{2}\} \)?

What is \( F \)?
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Expectations of continuous random variables

- Recall that when $X$ was a discrete random variable, with $p(x) = P\{X = x\}$, we wrote
  \[
  E[X] = \sum_{x : p(x) > 0} p(x)x.
  \]

- How should we define $E[X]$ when $X$ is a continuous random variable?
  
  **Answer:** $E[X] = \int_{-\infty}^{\infty} f(x)x\,dx$.

- Recall that when $X$ was a discrete random variable, with $p(x) = P\{X = x\}$, we wrote
  \[
  E[g(X)] = \sum_{x : p(x) > 0} p(x)g(x).
  \]

- What is the analog when $X$ is a continuous random variable?
  
  **Answer:** we will write $E[g(X)] = \int_{-\infty}^{\infty} f(x)g(x)\,dx$. 

18.440 Lecture 15
Variance of continuous random variables

- Suppose $X$ is a continuous random variable with mean $\mu$.
- We can write $\text{Var}[X] = E[(X - \mu)^2]$, same as in the discrete case.
- Next, if $g = g_1 + g_2$ then
  
  \[ E[g(X)] = \int g_1(x)f(x)\,dx + \int g_2(x)f(x)\,dx = \int (g_1(x) + g_2(x))f(x)\,dx = E[g_1(X)] + E[g_2(X)]. \]
- Furthermore, $E[ag(X)] = aE[g(X)]$ when $a$ is a constant.
- Just as in the discrete case, we can expand the variance expression as $\text{Var}[X] = E[X^2 - 2\mu X + \mu^2]$ and use additivity of expectation to say that
  
  \[ \text{Var}[X] = E[X^2] - 2\mu E[X] + E[\mu^2] = E[X^2] - 2\mu^2 + \mu^2 = E[X^2] - E[X]^2. \]
- This formula is often useful for calculations.
Examples

- Suppose that $f_X(x) = \begin{cases} 1/2 & x \in [0, 2] \\ 0 & x \notin [0, 2]. \end{cases}$

- What is $\text{Var}[X]$?

- Suppose instead that $f_X(x) = \begin{cases} x/2 & x \in [0, 2] \\ 0 & 0 \notin [0, 2]. \end{cases}$

- What is $\text{Var}[X]$?
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Measurable sets and a famous paradox
Uniform measure on $[0, 1]$

- One of the very simplest probability density functions is
  
  $$f(x) = \begin{cases} 
  1 & x \in [0, 1] \\
  0 & 0 \not\in [0, 1].
  \end{cases}$$

- If $B \subset [0, 1]$ is an interval, then $P\{X \in B\}$ is the length of that interval.

- Generally, if $B \subset [0, 1]$ then $P\{X \in B\} = \int_B 1dx = \int 1_B(x)dx$ is the “total volume” or “total length” of the set $B$.

- What if $B$ is the set of all rational numbers?

- How do we mathematically define the volume of an arbitrary set $B$?
Do all sets have probabilities? A famous paradox:

- Uniform probability measure on $[0, 1)$ should satisfy **translation invariance**: If $B$ and a horizontal translation of $B$ are both subsets $[0, 1)$, their probabilities should be equal.
- Consider **wrap-around translations** $\tau_r(x) = (x + r) \mod 1$.
- By translation invariance, $\tau_r(B)$ has same probability as $B$.
- Call $x, y$ “equivalent modulo rationals” if $x - y$ is rational (e.g., $x = \pi - 3$ and $y = \pi - 9/4$). An **equivalence class** is the set of points in $[0, 1)$ equivalent to some given point.
- There are uncountably many of these classes.
- Let $A \subset [0, 1)$ contain one point from each class. For each $x \in [0, 1)$, there is one $a \in A$ such that $r = x - a$ is rational.
- Then each $x$ in $[0, 1)$ lies in $\tau_r(A)$ for one rational $r \in [0, 1)$.
- Thus $[0, 1) = \bigcup \tau_r(A)$ as $r$ ranges over rationals in $[0, 1)$.
- If $P(A) = 0$, then $P(S) = \sum_r P(\tau_r(A)) = 0$. If $P(A) > 0$ then $P(S) = \sum_r P(\tau_r(A)) = \infty$. Contradicts $P(S) = 1$ axiom.
Three ways to get around this

1. **Re-examine axioms of mathematics:** the very *existence* of a set $A$ with one element from each equivalence class is consequence of so-called **axiom of choice**. Removing that axiom makes paradox goes away, since one can just suppose (pretend?) these kinds of sets don’t exist.

2. **Re-examine axioms of probability:** Replace *countable additivity* with *finite additivity*? (Look up Banach-Tarski.)

3. **Keep the axiom of choice and countable additivity but don’t define probabilities of all sets:** Instead of defining $P(B)$ for *every* subset $B$ of sample space, restrict attention to a family of so-called “measurable” sets.

Most mainstream probability and analysis takes the third approach.

In practice, sets we care about (e.g., countable unions of points and intervals) tend to be measurable.
More advanced courses in probability and analysis (such as 18.125 and 18.175) spend a significant amount of time rigorously constructing a class of so-called measurable sets and the so-called Lebesgue measure, which assigns a real number (a measure) to each of these sets.

These courses also replace the Riemann integral with the so-called Lebesgue integral.

We will not treat these topics any further in this course.

We usually limit our attention to probability density functions $f$ and sets $B$ for which the ordinary Riemann integral $\int 1_B(x)f(x)dx$ is well defined.

Riemann integration is a mathematically rigorous theory. It’s just not as robust as Lebesgue integration.