18.440: Lecture 18

Uniform random variables

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Uniform random variable on $[0, 1]$

Uniform random variable on $[\alpha, \beta]$

Motivation and examples
Uniform random variable on $[0, 1]$
Recall continuous random variable definitions

- Say $X$ is a **continuous random variable** if there exists a **probability density function** $f = f_X$ on $\mathbb{R}$ such that $P\{X \in B\} = \int_B f(x)dx := \int 1_B(x)f(x)dx$.

- We may assume $\int_{\mathbb{R}} f(x)dx = \int_{-\infty}^{\infty} f(x)dx = 1$ and $f$ is non-negative.

- Probability of interval $[a, b]$ is given by $\int_a^b f(x)dx$, the area under $f$ between $a$ and $b$.

- Probability of any single point is zero.

- Define **cumulative distribution function** $F(a) = F_X(a) := P\{X < a\} = P\{X \leq a\} = \int_{-\infty}^{a} f(x)dx$. 

18.440 Lecture 18
Uniform random variables on $[0, 1]$

- Suppose $X$ is a random variable with probability density function $f(x) = \begin{cases} 
1 & x \in [0, 1] \\
0 & x \notin [0, 1]. 
\end{cases}$

- Then for any $0 \leq a \leq b \leq 1$ we have $P\{X \in [a, b]\} = b - a$.

- Intuition: all locations along the interval $[0, 1]$ equally likely.

- Say that $X$ is a **uniform random variable on** $[0, 1]$ or that $X$ is **sampled uniformly from** $[0, 1]$. 
Suppose \( X \) is a random variable with probability density function \( f(x) = \begin{cases} 1 & x \in [0, 1] \\ 0 & x \not\in [0, 1] \end{cases} \).

What is \( E[X] \)?

Guess 1/2 (since 1/2 is, you know, in the middle).

Indeed, \( \int_{-\infty}^{\infty} f(x) x \, dx = \int_0^1 x \, dx = \frac{x^2}{2} \bigg|_0^1 = 1/2 \).

What would you guess the variance is? Expected square of distance from 1/2?

It’s obviously less than 1/4, but how much less?

\[
E[X^2] = \int_{-\infty}^{\infty} f(x) x^2 \, dx = \int_0^1 x^2 \, dx = \frac{x^3}{3} \bigg|_0^1 = 1/3.
\]

So \( \text{Var}[X] = E[X^2] - (E[X])^2 = 1/3 - 1/4 = 1/12 \).
Properties of uniform random variable on \([0, 1]\)

- Suppose \(X\) is a random variable with probability density function
  \[
  f(x) = \begin{cases} 
  1 & x \in [0, 1] \\
  0 & x \notin [0, 1].
  \end{cases}
  \]

- What is the cumulative distribution function \(F_X(a) = P\{X < a\}\)?
  \[
  F_X(a) = \begin{cases} 
  0 & a < 0 \\
  a & a \in [0, 1]. \\
  1 & a > 1.
  \end{cases}
  \]

- What is the general moment \(E[X^k]\) for \(k \geq 0\)?
Outline

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Uniform random variable on $[\alpha, \beta]$

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Motivation and examples
Uniform random variables on $[\alpha, \beta]$

- Fix $\alpha < \beta$ and suppose $X$ is a random variable with
  probability density function $f(x) = \begin{cases} \frac{1}{\beta - \alpha} & x \in [\alpha, \beta] \\ 0 & x \not\in [\alpha, \beta]. \end{cases}$

- Then for any $\alpha \leq a \leq b \leq \beta$ we have $P\{X \in [a, b]\} = \frac{b-a}{\beta - \alpha}$.

- Intuition: all locations along the interval $[\alpha, \beta]$ are equally likely.

- Say that $X$ is a uniform random variable on $[\alpha, \beta]$ or that $X$ is sampled uniformly from $[\alpha, \beta]$. 
Properties of uniform random variable on \([0, 1]\)

- Suppose \(X\) is a random variable with probability density function \(f(x) = \begin{cases} \frac{1}{\beta - \alpha} & x \in [\alpha, \beta] \\ 0 & x \notin [\alpha, \beta]. \end{cases}\)

- What is \(E[X]\)?
- Intuitively, we’d guess the midpoint \(\frac{\alpha + \beta}{2}\).
- What’s the cleanest way to prove this?
- One approach: let \(Y\) be uniform on \([0, 1]\) and try to show that \(X = (\beta - \alpha)Y + \alpha\) is uniform on \([\alpha, \beta]\).
- Then linearity of
  \[E[X] = (\beta - \alpha)E[Y] + \alpha = (1/2)(\beta - \alpha) + \alpha = \frac{\alpha + \beta}{2}.\]
- Using similar logic, what is the variance \(\text{Var}[X]\)?
- Answer: \(\text{Var}[X] = \text{Var}[(\beta - \alpha)Y + \alpha] = \text{Var}[(\beta - \alpha)Y] = (\beta - \alpha)^2 \text{Var}[Y] = (\beta - \alpha)^2 / 12.\)
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Motivation and examples
Uniform random variables and percentiles

- Toss \( n = 300 \) million Americans into a hat and pull one out uniformly at random.
- Is the height of the person you choose a uniform random variable?
- Maybe in an approximate sense?
- No.
- Is the \textit{percentile} of the person I choose uniformly random? In other words, let \( p \) be the fraction of people left in the hat whose heights are less than that of the person I choose. Is \( p \), in some approximate sense, a uniform random variable on \([0, 1]\)?
- The way I defined it, \( p \) is uniform from the set \( \{0, 1/(n - 1), 2/(n - 1), \ldots, (n - 2)/(n - 1), 1\} \). When \( n \) is large, this is kind of like a uniform random variable on \([0, 1]\).
Approximately uniform random variables

- Intuition: which of the following should give approximately uniform random variables?
- 1. Toss \( n = 300 \) million Americans into a hat, pull one out uniformly at random, and consider that person’s height (in centimeters) modulo one.
- 2. The location of the first raindrop to land on a telephone wire stretched taut between two poles.
- 3. The amount of time you have to wait until the next subway train come (assuming trains come promptly every six minutes and you show up at kind of a random time).
- 4. The amount of time you have to wait until the next subway train (without the parenthetical assumption above).
Approximately uniform random variables

5. How about the location of the jump between times 0 and 1 of $\lambda$-Poisson point process (which we condition to have exactly one jump between $[0, 1]$)?

6. The location of the ace of spades within a shuffled deck of 52 cards.
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