1. Problem 6.4.1

\(Z \sim N(0, 1)\) and \(U \sim \chi^2_n\) and \(Z\) and \(U\) are independent.

\[T = \frac{Z}{\sqrt{U/n}}\] a Student’s \(t\) random variable with \(n\) degrees of freedom.

Find the density function of \(T\).

**Solution:**

- The density of \(Z\) is \(f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}, -\infty < x < +\infty\).
- The density of \(U\) is
  \[
f_U(u) = \begin{cases} 
  \frac{1}{\Gamma(n/2)2^{n/2}} u^{(n/2)-1}e^{-u/2}, & \text{if } u > 0 \\
  0, & \text{if } u \leq 0
  \end{cases}
  \]
- Because \(Z\) and \(U\) are independent their joint density is
  \(f_{Z,U}(z, u) = f_Z(z)f_U(u)\)
- Consider transforming \((Z, U)\) to \((T, V)\), where
  \[T = \frac{Z}{\sqrt{U/n}}\] and \(V = U\),
  computing the joint density of \((T, V)\) and then integrating out \(V\) to obtain the marginal density of \(T\).
  
- Determine the functions \(g(T, V) = Z\) and \(h(T, V) = U\)

  \[g(T, V) = \sqrt{V/n}T\]
  \[h(T, V) = V\]

  Then the joint density of \((T, V)\) is given by
  \(f_{T,V}(t, v) = f_{Z,U}(g(t, v), h(t, v)) \times J\)
  where \(J\) is the Jacobian of the transformation from \((Z, U)\) to \((T, V)\).

  Compute \(J\):

  \[
  J = \begin{vmatrix} 
  \frac{\partial g(t, v)}{\partial t} & \frac{\partial g(t, v)}{\partial v} \\
  \frac{\partial h(t, v)}{\partial t} & \frac{\partial h(t, v)}{\partial v}
  \end{vmatrix}
  = \begin{vmatrix} 
  \sqrt{V/n} & (T/n)^{1/2}V^{-1/2} \\
  0 & 1
  \end{vmatrix}
  = \sqrt{V/n}
  \]
The joint density of \((T, V)\) is thus
\[
f_{T,V}(t, v) = f_Z(g(t, v)) \times f_U(h(t, v)) \times J
\]
\[
= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(v/\sqrt{n})^2} \times \frac{1}{\Gamma(n/2)2^{n/2}} v^{(n/2)-1}e^{-v/2} \times \sqrt{v/n}
\]
\[
= \frac{1}{\sqrt{2\pi n}\Gamma(n/2)2^{n/2}} v^{\frac{1}{2}(n+1)-1}e^{-\frac{v}{2}(1+\frac{t^2}{n})} J
\]

Integrate over \(v\) to obtain the marginal density of \(T\):
\[
f_T(t) = \int_0^\infty f_{T,V}(t, v) dv
\]
\[
= \frac{1}{\sqrt{2\pi n}\Gamma(n/2)2^{n/2}} \int_0^\infty v^{\frac{1}{2}(n+1)-1}e^{-\frac{v}{2}(1+\frac{t^2}{n})} dv
\]

The integral factor can be evaluated by recognizing that it is identical to integrating a \(Gamma(\alpha, \lambda)\) density function apart from the normalization constant, with \(\alpha = \frac{(n+1)}{2}\) and \(\lambda = \frac{1}{2}(1+\frac{t^2}{n})\), that is:
\[
\Gamma(\alpha)\lambda^{-\alpha} = \int_0^\infty \lambda^{\alpha-1}e^{-\lambda v} dv
\]
which gives
\[
\int_0^\infty v^{\frac{1}{2}(n+1)-1}e^{-\frac{v}{2}(1+\frac{t^2}{n})} dv = \Gamma((n+1)/2) \times \frac{1}{2}(1+\frac{t^2}{n})]^{-(n+1)/2}
\]

Finally we can write
\[
f_T(t) = \int_0^\infty f_{T,V}(t, v) dv
\]
\[
= \frac{1}{\sqrt{2\pi n}\Gamma(n/2)2^{n/2}} \int_0^\infty v^{\frac{1}{2}(n+1)-1}e^{-\frac{v}{2}(1+\frac{t^2}{n})} dv
\]
\[
= \frac{1}{\sqrt{2\pi n}\Gamma(n/2)2^{n/2}} \Gamma((n+1)/2) \times \frac{1}{2}(1+\frac{t^2}{n})]^{-(n+1)/2}
\]
\[
= \frac{\Gamma((n+1)/2)}{\sqrt{\pi n}\Gamma(n/2)} \times \left[1 + \frac{t^2}{n}\right]^{-(n+1)/2}
\]

2. Suppose the random variable \(X\) has a \(t\) distribution with \(n\) degrees of freedom.

(a). For what values of \(n\) is the variance finite/infinite.

(b). Derive a formula for the variance of \(X\) (when it is finite).

Solution:

(a). For the variance of the \(t\) distribution to be finite it must have a finite second moment:
The integrand of this second moment calculation is proportional to

\[ t^2 f_T(t) \propto \left[ \frac{t^2}{1 + \frac{n}{n+1}} \right]^{(n+1)/2} \xrightarrow{t \to \infty} n^{(n+1)/2} \times t^{2-(n+1)} \propto t^{1-n} \]

The integral of this integrand thus converges if and only if

\[(1 - n) < (-1), \text{ which is equivalent to } n > 2.\]

(b). If \(n > 2\), then the variance of \(T\) is finite. For such \(n\), the mean of \(T\) exists and is zero, so writing \(T = Z/\sqrt{U/n}\) for independent \(Z \sim N(0,1)\) and \(U \sim \chi^2_n\)

\[
Var(T) = E[T^2] = E[Z/\sqrt{U/n}]^2 = E[nZ^2/U]
\]

\[
= n \times E[Z^2] \times E[\frac{1}{U}]
\]

\[
= \frac{1 \times n \times \frac{1}{n-2}}{n-2}
\]

The expectation \(E[\frac{1}{U}] = 1/(n-2)\) can be computed directly for \(n > 2\).

(Note that the formula is undefined for \(n = 2\) and gives negative values for \(n < 2\))

3. 6.4.4. Also, add part (c) answer the question if the random variable \(T\) follows a standard normal distribution \(N(0, 1)\). Comment on the differences and why that should be.

**Solution:**

We are given that \(T\) follows a \(t_7\) distribution. The problem is solved by finding an expression for \(t_0\) in terms of the cumulative distribution function of \(T\).

(a). To find the \(t_0\) such that \(P(|T| < t_0) = .9\) this is equivalent to \(P(T < .95)\), which is solved in R using the function \(qt()\) – the quantile function for the \(t\) distribution

> \texttt{args(qt)}

function \texttt{(p, df, ncp, lower.tail = TRUE, log.p = FALSE)}

> \texttt{qt(.95, df=7)}

[1] 1.894579
So, $t_0 = 1.894579$.

(b). $P(T > t_0) = .05$ is equivalent to $P(T \leq t_0) = 1 - .05 = .95$. This is the same $t_0$ found in (a).

> args(qt)
function (p, df, ncp, lower.tail = TRUE, log.p = FALSE)

> qt(p=.95, df=7)
[1] 1.894579
# Which is equivalent to
> qt(p=.05, df=7, lower.tail=FALSE)
[1] 1.894579

(c). For the standard normal distribution we use qnorm() – the quantile function for the $\text{Normal}(0, 1)$ distribution

> args(qnorm)
function (p, mean = 0, sd = 1, lower.tail = TRUE, log.p = FALSE)
NULL
> qnorm(.95)
[1] 1.644854

> qnorm(p=.05, lower.tail=FALSE)
[1] 1.644854

So for both parts (a) and (b) $t_0 = 1.644854$ for the $N(0, 1)$ r.v. versus $t_0 = 1.894579$ for the $t$ distribution with 7 degrees of freedom.

The $t_0$ values are larger for the $t$ distribution indicating that the $t$ distribution has heavier tail areas than the $\text{Normal}(0, 1)$ distribution. This makes sense because the $t$ distribution equals a $\text{Normal}(0,1)$ random variable divided by a random variable with expectation equal to 1 but positive variance. The possibility of the denominator of the $t$ ratio being less than 1 increases the probability of larger values.

4. Problem 8.10.10.

Use the normal approximation of the Poisson distribution to sketch the approximate sampling distribution of $\hat{\lambda}$ of Example A of Section 8.4. According to this approximation, what is
\[ P(|\lambda_0 - \hat{\lambda}| > \delta) \text{ for } \delta = -.5, 1, 1.5, 2, 2.5 \]

where \( \lambda_0 \) is the true value of \( \lambda \).

**Solution:**

In the example, the estimate \( \hat{\lambda} = \bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i = 24.9 \), with \( n = 23 \).

The \( X_1, \ldots, X_n \) are assumed to be i.i.d. (independent and identically distributed) \( \text{Poisson}(\lambda_0) \) random variables with

\[ E[X_i] = \lambda_0 \text{ and } \text{Var}[X_i] = \lambda_0 \]

By the Central Limit Theorem

\[ \sqrt{n}(\bar{X} - \lambda_0) \xrightarrow{n \to \infty} N(0, 1) \]

The approximate sampling distribution of \( \hat{\lambda} \) is thus a Normal distribution centered \( \lambda_0 \) with standard deviation equal to \( \sqrt{\lambda_0/n} \approx \sqrt{\frac{\lambda}{1.040485}} = 24.9/23 = 1.04085 \).

For the probability computations:

\[ P(|\lambda_0 - \hat{\lambda}| > \delta) = P(|\lambda_0 - \bar{X}| > \delta) \]

\[ = P\left( \sqrt{n} \frac{|\lambda_0 - \bar{X}|}{\sqrt{\lambda_0}} > \sqrt{n} \frac{\delta}{\sqrt{\lambda_0}} \right) \]

\[ \approx P(|N(0, 1)| > \sqrt{n} \frac{\delta}{\sqrt{\lambda_0}}) \]

\[ = P(|N(0, 1)| > \sqrt{\frac{23}{24.9}} \delta) \]

\[ = 2 \times (1 - \Phi(\sqrt{\frac{23}{24.9}} \times \delta)) \]

Using R and the function \( \text{pnorm} \) we can compute the desired values:

\[ > 2 \times 1 - \text{pnorm}\left( \text{sqrt}(23/24.9) \times (1.5, 1.5, 2, 2.5) \right) \]

[1] 1.315420 1.168253 1.074703 1.027292 1.008137
5. Problem 8.10.13.

In Example D of Section 8.4, the method of moments estimate was found to be \( \hat{\alpha} = 3 \overline{X} \). In this example, consider the sampling distribution of \( \hat{\alpha} \).

(a). Show that \( E(\hat{\alpha}) = \alpha \), that is, that the estimate is unbiased.

(b). Show that \( Var(\hat{\alpha}) = (3 - \alpha^2)/n \).

(c). Use the central limit theorem to deduce a normal approximation to the sampling distribution of \( \hat{\alpha} \).

According to this approximation, if \( n = 25 \) and \( \alpha = 0 \), what is the \( P(|\hat{\alpha}| > .5) \).

**Solution**

The sample of values \( X_1, \ldots, X_n \) giving \( \overline{X} \) are i.i.d. with density function

\[
f(x \mid \alpha) = \frac{1 + \alpha x}{2}, \text{ for } -1 \leq x \leq +1,
\]

with parameter \( \alpha : -1 \leq \alpha \leq 1 \). (The values are such that \( x_i = \cos(\theta_i) \), where \( \theta_i \) is the angle at which electrons are emitted in muon decay.)

(a). Since the \( X_i \) are i.i.d.

\[
E[\overline{X}] = E[X_i] = \int_{-1}^{1} x \times \left( \frac{1 + \alpha x}{2} \right) dx = \alpha/3
\]

It follows that

\[
E[\hat{\alpha}] = E[3\overline{X}] = 3E[\overline{X}] = 3(\alpha/3) = \alpha
\]

(b). Since the \( X_i \) are i.i.d.

\[
Var[\overline{X}] = Var[X_i]/n = \left( \frac{1}{n} \right) \times (E[X^2] - E[X]^2)
\]

\[
= \left( \frac{1}{n} \right) \times \left( \left[ \int_{-1}^{1} x^2 \times \left( \frac{1 + \alpha x}{2} \right) dx \right] - (\alpha/3)^2 \right)
\]

\[
= \left( \frac{1}{n} \right) \times \left( \left[ \frac{1}{2} \int_{-1}^{1} x^2 dx \right] - (\alpha/3)^2 \right)
\]

\[
= \left( \frac{1}{n} \right) \times \left( \left[ \frac{1}{3} \right] - (\alpha/3)^2 \right)
\]

\[
= \frac{3 - \alpha^2}{9n}
\]

It follows that:

\[
Var[\hat{\alpha}] = Var[3\overline{X}] = 9 \times Var[\overline{X}] = \frac{3 - \alpha^2}{n}.
\]

(c) By the central limit theorem, for true parameter \( \alpha = \alpha_0 \), it follows that

\[
\hat{\alpha} \xrightarrow{n \to \infty} N(\alpha_0, \frac{3 - \alpha_0^2}{n})
\]