Hao Wu

MIT

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Recall Suppose that $P$ is irreducible.

- The Markov chain is recurrent if and only if
  $$\mathbb{P}_x[\tau_x^+ < \infty] = 1, \quad \text{for some } x.$$  

- The Markov chain is positive recurrent if and only if
  $$\mathbb{E}_x[\tau_x^+] < \infty, \quad \text{for some } x.$$  

Today’s Goal

- stationary distribution
- convergence to stationary distribution
Stationary distribution

Theorem

An irreducible Markov chain is positive recurrent if and only if there exists a probability measure $\pi$ on $\Omega$ such that $\pi = \pi P$.

Corollary

If an irreducible Markov chain is positive recurrent, then

- there exists a probability measure $\pi$ such that $\pi = \pi P$;
- $\pi(x) > 0$ for all $x$. In fact,

$$\pi(x) = \frac{1}{\mathbb{E}_x[\tau_x^+]}.$$
Convergence to the stationary

Theorem

If an irreducible Markov chain is positive recurrent and aperiodic, then

$$\lim_{n} P_x[X_n = y] = \pi(y) > 0, \quad \text{for all } x, y.$$ 

Theorem

If an irreducible Markov chain is null recurrent, then

$$\lim_{n} P_x[X_n = y] = 0, \quad \text{for all } x, y.$$
**Recall** Consider a Markov chain with state space $\Omega$ (countable) and transition matrix $P$. For each $x \in \Omega$, define

$$T(x) = \{n \geq 1 : P^n(x, x) > 0\}.$$ 

Then

$$gcd(T(x)) = gcd(T(y)),$$ 

for all $x, y$.

We say the chain is aperiodic if $gcd(T(x)) = 1$.

**Theorem**

**Suppose that the Markov chain is irreducible and aperiodic. If the chain is positive recurrent, then**

$$\lim_{n \to \infty} ||P^n(x, \cdot) - \pi||_{TV} = 0.$$
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