Lecture 15: Introduction to martingales

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About the midterm: total = 23

1 in [80, 100], 5 in [70, 80), 6 in [60, 70)
4 in [40, 60), 7 in [10, 40)

Today’s Goal:
- probability space
- conditional expectation
- introduction to martingales
Probability space

Definition

\( \Omega \) : a set. A collection \( \mathcal{F} \) of subsets of \( \Omega \) is called a \( \sigma \)-algebra on \( \Omega \) if

- \( \Omega \in \mathcal{F} \)
- \( F \in \mathcal{F} \implies F^c \in \mathcal{F} \)
- \( F_1, F_2, \ldots \in \mathcal{F} \implies \bigcup_n F_n \in \mathcal{F} \).

The pair \((\Omega, \mathcal{F})\) is called a measurable space.

Definition

Let \((\Omega, \mathcal{F})\) be a measurable space. A map \( \mathbb{P} : \mathcal{F} \to [0, 1] \) is called a probability measure if

- \( \mathbb{P}[\emptyset] = 0, \mathbb{P}[\Omega] = 1 \)
- it is countably additive: whenever \((F_n)_{n \geq 0}\) is a sequence of disjoint sets in \( \Omega \), then \( \mathbb{P}[\bigcup n F_n] = \sum_n \mathbb{P}[F_n] \).
Probability space

$(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space
- $\Omega$ : state space
- $\mathcal{F}$ : $\sigma$-algebra
- $\mathbb{P}$ : probability measure
Conditional expectation—motivation

- \((\Omega, \mathcal{F}, \mathbb{P})\) a probability space
- \(X, Z\) two random variables
- elementary conditional probability:
  \[ P[X = x \mid Z = z] = \frac{P[X = x, Z = z]}{P[Z = z]} \]
- elementary conditional expectation:
  \[ \mathbb{E}[X \mid Z = z] = \sum_x xP[X = x \mid Z = z] \]
- \(Y = \mathbb{E}[X \mid \sigma(Z)]\)?
  - \(Y\) is measurable with respect to \(\sigma(Z)\)
  - \(\mathbb{E}[Y 1_{Z=z}] = \mathbb{E}[X 1_{Z=z}]\)
Conditional Expectation

- \((\Omega, \mathcal{F}, \mathbb{P})\) is a probability space
- \(X\) is a random variable on the probability space with \(\mathbb{E}[|X|] < \infty\)
- \(A \subset \mathcal{F}\) is a sub \(\sigma\)-algebra

Then there exists a random variable \(Y\) such that
- \(Y\) is \(A\)-measurable with \(\mathbb{E}[|Y|] < \infty\)
- for any \(A \in A\), we have \(\mathbb{E}[Y1_A] = \mathbb{E}[X1_A]\).

Moreover, if \(\tilde{Y}\) also satisfies the above two properties, then \(\tilde{Y} = Y\) a.s.

A random variable \(Y\) with the above two properties is called the **conditional expectation** of \(X\) given \(A\), and we denote it by \(\mathbb{E}[X \mid A]\).

**Remark:**
- If \(A = \{\emptyset, \Omega\}\), then \(\mathbb{E}[X \mid A] = \mathbb{E}[X]\).
- If \(X\) is \(A\)-measurable, then \(\mathbb{E}[X \mid A] = X\).
- If \(Y = \mathbb{E}[X \mid A]\), then \(\mathbb{E}[Y] = \mathbb{E}[X]\).
Conditional Expectation—Basic properties

Suppose that \((\Omega, \mathcal{F}, \mathbb{P})\) is a probability space and that
- \(X, X_n\) are random variables on the probability space in \(L^1\)
- \(\mathcal{A} \subset \mathcal{F}\) is a sub \(\sigma\)-algebra

Then we have the following.

- **(Linearity)** \(\mathbb{E}[a_1 X_1 + a_2 X_2 \mid \mathcal{A}] = a_1 \mathbb{E}[X_1 \mid \mathcal{A}] + a_2 \mathbb{E}[X_2 \mid \mathcal{A}]\) for constants \(a_1, a_2\).

- **(Positivity)** If \(X \geq 0\) a.s., then \(\mathbb{E}[X \mid \mathcal{A}] \geq 0\) a.s.

- **(Monotone convergence)** If \(0 \leq X_n \uparrow X\) a.s. then \(\mathbb{E}[X_n \mid \mathcal{A}] \uparrow \mathbb{E}[X \mid \mathcal{A}]\) a.s.

- **(Fatou’s Lemma)** If \(X_n \geq 0\), then
  \(\mathbb{E}[\lim \inf_n X_n \mid \mathcal{A}] \leq \lim \inf_n \mathbb{E}[X_n \mid \mathcal{A}]\) a.s.

- **(Dominated convergence)** If \(|X_n| \leq Z\) with \(Z \in L^1\) and \(X_n \to X\) a.s., then \(\mathbb{E}[X_n \mid \mathcal{A}] \to \mathbb{E}[X \mid \mathcal{A}]\) a.s.

- **(Jensen inequality)** If \(\varphi : \mathbb{R} \to \mathbb{R}\) is convex and \(\mathbb{E}[|\varphi(X)|] < \infty\), then \(\mathbb{E}[\varphi(X) \mid \mathcal{A}] \geq \varphi(\mathbb{E}[X \mid \mathcal{A}])\).
Suppose that \((\Omega, \mathcal{F}, \mathbb{P})\) is a probability space and that

- \(X, X_n\) are random variables on the probability space in \(L^1\)
- \(\mathcal{A} \subset \mathcal{F}\) is a sub \(\sigma\)-algebra

Then we have the following.

- **(Tower property)** If \(\mathcal{B}\) is a sub-\(\sigma\)-algebra of \(\mathcal{A}\), then
  \[
  \mathbb{E}\left[ \mathbb{E}[X | \mathcal{A}] | \mathcal{B} \right] = \mathbb{E}[X | \mathcal{B}] \text{ a.s.}
  \]

- **("Taking out what is known")** If \(Z\) is \(\mathcal{A}\)-measurable and bounded, then
  \[
  \mathbb{E}[XZ | \mathcal{A}] = Z\mathbb{E}[X | \mathcal{A}] \text{ a.s.}
  \]

- **(Independence)** If \(\mathcal{B}\) is independent of \(\sigma(\sigma(X), \mathcal{A})\), then
  \[
  \mathbb{E}[X | \sigma(\mathcal{A}, \mathcal{B})] = \mathbb{E}[X | \mathcal{A}] \text{ a.s.}
  \] In particular, if \(X\) is independent of \(\mathcal{B}\), then
  \[
  \mathbb{E}[X | \mathcal{B}] = \mathbb{E}[X] \text{ a.s.}
  \]
Conditional expectation—example

Suppose that \((X_n)_{n \geq 0}\) are i.i.d. with the same distribution as \(X\) with \(E[|X|] < \infty\). Let \(S_n = X_1 + X_2 + \cdots + X_n\), and define

\[ A_n = \sigma(S_n, S_{n+1}, \ldots) = \sigma(S_n, X_{n+1}, \ldots). \]

**Question**: \(E[X_1 | A_n]\)?

**Answer**: \(E[X_1 | A_n] = S_n/n\).
Martingales

$(\Omega, \mathcal{F}, \mathbb{P})$ a probability space
A filtration $(\mathcal{F}_n)_{n \geq 0}$ is an increasing family of sub $\sigma$-algebras of $\mathcal{F}$.
A sequence of random variables $X = (X_n)_{n \geq 0}$ is adapted to $(\mathcal{F}_n)_{n \geq 0}$ if $X_n$ is measurable with respect to $\mathcal{F}_n$ for all $n$.

Let $(X_n)_{n \geq 0}$ be a sequence of random variables.
The natural filtration $(\mathcal{F}_n)_{n \geq 0}$ associated to $(X_n)_{n \geq 0}$ is given by
\[
\mathcal{F}_n = \sigma(X_k, k \leq n).
\]
We say that $(X_n)_{n \geq 0}$ is integrable if $X_n$ is integrable for all $n$.

Definition

Let $X = (X_n)_{n \geq 0}$ be an integrable process.
- $X$ is a martingale if $\mathbb{E}[X_n | \mathcal{F}_m] = X_m$ a.s. for all $n \geq m$.
- $X$ is a supermartingale if $\mathbb{E}[X_n | \mathcal{F}_m] \leq X_m$ a.s. for all $n \geq m$.
- $X$ is a submartingale if $\mathbb{E}[X_n | \mathcal{F}_m] \geq X_m$ a.s. for all $n \geq m$. 
Examples

**Example 1** Let \((\xi_i)_{i \geq 1}\) be i.i.d with \(\mathbb{E}[\xi_1] = 0\). Then \(X_n = \sum_{1}^{n} \xi_i\) is a martingale.

**Example 2** Let \((\xi_i)_{i \geq 1}\) be i.i.d with \(\mathbb{E}[\xi_1] = 1\). Then \(X_n = \prod_{1}^{n} \xi_i\) is a martingale.

**Example 3** Consider biased gambler’s ruin: at each step, the gambler gains one dollar with probability \(p\) and losses one dollar with probability \((1 - p)\). Let \(X_n\) be the money in purse at time \(n\).

- If \(p = 1/2\), then \((X_n)\) is a martingale.
- If \(p < 1/2\), then \((X_n)\) is a supermartingale.
- If \(p > 1/2\), then \((X_n)\) is a submartingale.
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