18.445 Introduction to Stochastic Processes
Lecture 17: Martingle: a.s convergence and $L^p$-convergence

Hao Wu
MIT
15 April 2015
Recall

- Martingale: $\mathbb{E}[X_n | \mathcal{F}_m] = X_m$ for $n \geq m$.
- Optional Stopping Theorem: $\mathbb{E}[X_T] = \mathbb{E}[X_0]$?

Today’s goal

- a.s.martingale convergence
- Doob’s maximal inequality
- convergence in $L^p$ for $p > 1$
Various convergences

Spaces
- $L^1$ space: $\mathbb{E}[|X|] < \infty$.
  - $L^1$-norm: $||X||_1 = \mathbb{E}[|X|]$.
  - triangle inequality: $||X + Y||_1 \leq ||X||_1 + ||Y||_1$.
- $L^p$ space for $p > 1$: $\mathbb{E}[|X|^p] < \infty$
  - $L^p$-norm: $||X||_p = \mathbb{E}[|X|^p]^{1/p}$.
  - triangle inequality: $||X + Y||_p \leq ||X||_p + ||Y||_p$.

Lemma

For $p > 1$, $L^p$ is contained in $L^1$.

different notions of convergence
- almost sure convergence: $X_n \rightarrow X_\infty$ a.s.
- convergence in $L^p$: $X_n \rightarrow X_\infty$ in $L^p$.
- convergence in $L^1$: $X_n \rightarrow X_\infty$ in $L^1$. 
A.S. Martingale Convergence

**Theorem**

Let $X = (X_n)_{n \geq 0}$ be a supermartingale which is bounded in $L^1$, i.e.

$$\sup_n \mathbb{E}[|X_n|] < \infty.$$  

Then

$$X_n \to X_\infty, \quad \text{almost surely, as} \quad n \to \infty,$$

for some $X_\infty \in L^1$.

**Proof** Attached on the website.

**Corollary**

Let $X = (X_n)_{n \geq 0}$ be a non-negative supermartingale. Then $X_n$ converges a.s. to some a.s. finite limit.
Examples

Example 1 Let \((\xi_j)_{j \geq 1}\) be independent random variables with mean zero such that \(\sum_{j=1}^{\infty} \mathbb{E}[|\xi_j|] < \infty\). Set

\[ X_0 = 0, \quad X_n = \sum_{j=1}^{n} \xi_j. \]

- \((X_n)_{n \geq 0}\) is a martingale bounded in \(L^1\).
- \(X_n\) converges a.s. to \(X_\infty = \sum_{j=1}^{\infty} \xi_j\).
- In fact, \(X_n\) also converges to \(X_\infty\) in \(L^1\).

Example 2 Let \((\xi_j)_{j \geq 1}\) be non-negative independent random variables with mean one. Set

\[ X_0 = 1, \quad X_n = \prod_{j=1}^{n} \xi_j. \]

- \((X_n)_{n \geq 0}\) is a non-negative martingale.
- \(X_n\) converges a.s. to some limit \(X_\infty \in L^1\).
Question

Suppose that a martingale $X$ is bounded in $L^1$, then we have the a.s. convergence.

**Question** : Do we have $\mathbb{E}[X_\infty] = \mathbb{E}[X_0]$?

**Answer** : It is true when we have convergence in $L^1$.

- Convergence in $L^p$ for $p > 1$ implies convergence in $L^1$. (Today)
- Convergence in $L^1$. (Next lecture)
Doob’s maximal inequality

**Theorem**

Let \( X = (X_n)_{n \geq 0} \) be a non-negative submartingale. Define \( X^*_n = \max_{0 \leq k \leq n} X_k \). Then

\[
\lambda \mathbb{P}[X^*_n \geq \lambda] \leq \mathbb{E}[X_n 1[X^*_n \geq \lambda]] \leq \mathbb{E}[X_n].
\]

**Theorem**

Let \( X = (X_n)_{n \geq 0} \) be a non-negative submartingale. Define \( X^*_n = \max_{0 \leq k \leq n} X_k \). Then, for all \( p > 1 \), we have

\[
\|X^*_n\|_p \leq \frac{p}{p - 1} \|X_n\|_p.
\]

**Recall** Hölder inequality: \( p > 1, q > 1 \) and \( 1/p + 1/q = 1 \), then

\[
\mathbb{E}[|XY|] \leq \mathbb{E}[|X|^p]^{1/p} \times \mathbb{E}[|Y|^q]^{1/q}.
\]
**Theorem**

Let $X = (X_n)_{n \geq 0}$ be a martingale and $p > 1$, then the following statements are equivalent.

1. $X$ is bounded in $L^p$: $\sup_{n \geq 0} \|X_n\|_p < \infty$
2. $X$ converges a.s and in $L^p$ to a random variable $X_\infty$.
3. There exists a random variable $Z \in L^p$ such that
   \[ X_n = \mathbb{E}[Z | \mathcal{F}_n] \quad \text{a.s.} \]

**Corollary**

Let $Z \in L^p$. Then

\[ \mathbb{E}[Z | \mathcal{F}_n] \to \mathbb{E}[Z | \mathcal{F}_\infty], \quad \text{a.s. and in } L^p. \]
Example

Let \((\xi_j)_{j \geq 1}\) be independent random variables with mean zero such that
\[
\sum_{j=1}^{\infty} \mathbb{E}[\xi_j^2] < \infty.
\]
Set
\[
X_0 = 0, \quad X_n = \sum_{j=1}^{n} \xi_j.
\]

- \((X_n)_{n \geq 0}\) is a martingale bounded in \(L^2\).
- \(X_n\) converges to \(X_\infty = \sum_{j=1}^{\infty} \xi_j\) a.s. and in \(L^2\).
- \(\mathbb{E}[X_\infty^2] = \sum_{j=1}^{\infty} \mathbb{E}[\xi_j^2]\).
Example

Let \((\xi_j)_{j \geq 1}\) be non-negative independent random variables with mean one. Set

\[X_0 = 1, \quad X_n = \prod_{j=1}^{n} \xi_j.\]

1. \((X_n)_{n \geq 0}\) is a non-negative martingale.
2. \(X_n\) converges a.s. to some limit \(X_\infty \in L^1\).

Question:

1. Do we have \(\mathbb{E}[X_\infty] = 1\) ?

Answer: Set \(a_j = \mathbb{E}[\sqrt{\xi_j}] \in (0, 1]\).

1. If \(\prod_j a_j > 0\), then \(X\) converges in \(L^1\) and \(\mathbb{E}[X_\infty] = 1\). (Next lecture)
2. If \(\prod_j a_j = 0\), then \(X_\infty = 0\) a.s.
18.445 Introduction to Stochastic Processes
Spring 2015

For information about citing these materials or our Terms of Use, visit: http://ocw.mit.edu/terms.