Recall: We consider continuous time Markov chain on countable state space with the following requirement

- (Homogeneity) $\mathbb{P}[X_{t+s} = y \mid X_s = x] = P_t(x, y)$
- (Right-continuity for the chain) For any $t \geq 0$, there exists $\epsilon > 0$, such that $X_{t+s} = X_t$ for all $s \in [0, \epsilon]$

Today’s Goal:

- More words about the regularity of continuous time Markov chain
- Infinitesimal generator
Consider a continuous time Markov chain \((X_t)_{t \geq 0}\).
Define the **jump times** of the chain : \(J_0, J_1, J_2, \ldots\)

\[
J_0 = 0, \quad J_{n+1} = \inf\{t > J_n : X_t \neq X_{J_n}\}, \quad n \geq 0.
\]

Define the **holding times** of the chain : \(S_1, S_2, \ldots\)

\[
S_n = J_n - J_{n-1}, \quad n \geq 1.
\]

Define the jump process of the chain : \(Y_0, Y_1, \ldots\)

\[
Y_n = X_{J_n}, \quad n \geq 0.
\]

- By right-continuity, we have \(S_n > 0\).
- If \(J_{n+1} = \infty\) for some \(n\), set \(X_\infty = X_{J_n}\)

**Example** Let \((X_t)_{t \geq 0}\) be a Poisson process. Then the jump process : \(Y_n = n\)
the holding times : \((S_n)_{n \geq 1}\) are i.i.d exponential.
Define the **explosion time** $\xi$ by

$$\xi = \sup_n J_n = \sum_n S_n.$$ 

We only consider the chains with $\xi = \infty$.

**Summary** We consider continuous time Markov chain on countable state space with the following requirement

- (Homogeneity) $\mathbb{P}[X_{t+s} = y \mid X_s = x] = P_t(x, y)$
- (Right-continuity for the chain) For any $t \geq 0$, there exists $\epsilon > 0$, such that $X_{t+s} = X_t$ for all $s \in [0, \epsilon]$
- (Non explosion) The explosion time $\xi = \infty$
Continuous time Markov chain

**Summary** We consider continuous time Markov chain on countable state space with the following requirement

- (Homogeneity) $\mathbb{P}[X_{t+s} = y \mid X_s = x] = P_t(x, y)$
- (Right-continuity for the chain) For any $t \geq 0$, there exists $\epsilon > 0$, such that $X_{t+s} = X_t$ for all $s \in [0, \epsilon]$
- (Non explosion) The explosion time $\xi = \infty$
- (Right-continuity in the semigroup) $P_\epsilon \to P_0 = I$ as $\epsilon \to 0$, pointwise for each entry.

Consider the transition semigroup $(P_t)_{t \geq 0}$

- $P_0 = I$
- $P_t$ is stochastic for all $t \geq 0$
- $P_{t+s} = P_t P_s$
- $P_\epsilon \to P_0 = I$ as $\epsilon \downarrow 0$

**Remark** Combining (3) and (4), the semigroup is right continuous for all $t$. 

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Infinitesimal generator

**Theorem**

Let \((P_t)_{t \geq 0}\) be a right-continuous transition semigroup.

- For any state \(x\), the limit exists
  \[
  q_x = \lim_{\epsilon \downarrow 0} \frac{(1 - P_\epsilon(x, x))}{\epsilon} \geq 0.
  \]

- For any distinct states \(x, y\), the limit exists
  \[
  q_{xy} = \lim_{\epsilon \downarrow 0} \frac{P_\epsilon(x, y)}{\epsilon} \geq 0.
  \]

**Lemma**

Let \(f : (0, \infty) \to \mathbb{R}\) be a nonnegative function such that \(\lim_{\epsilon \downarrow 0} f(\epsilon) = 0\), and assume that \(f\) is subadditive, that is,
\[
f(t + s) \leq f(t) + f(s), \quad \forall t, s \geq 0.
\]

Then the limit \(\lim_{\epsilon \downarrow 0} f(\epsilon)/\epsilon\) exists and equals \(\sup_{t > 0} f(t)/t\).
Infinitesimal generator

Definition

Set

\[ q_{xx} = -q_x = \lim_{\epsilon \downarrow 0} \frac{P_\epsilon(x, x) - 1}{\epsilon}, \quad q_{xy} = \lim_{\epsilon \downarrow 0} \frac{P_\epsilon(x, y)}{\epsilon}. \]

Then the matrix \( A = (q_{xy})_{x, y \in \Omega} \) is called the infinitesimal generator of the semigroup.

- \( q_{xx} \leq 0 \)
- \( q_{xy} \geq 0 \) for \( y \neq x \)
- \( \sum_y q_{xy} = 0 \)
Examples

Example 1 Let \((X_t)_{t \geq 0}\) be the Poisson process with intensity \(\lambda > 0\). Then
\[
q_{ii} = -\lambda, \quad q_{i,i+1} = \lambda.
\]

Example 2 Let \((\hat{X}_n)_{n \geq 0}\) be a discrete time Markov chain with transition matrix \(Q\). Let \((N_t)_{t \geq 0}\) be an independent Poisson process with intensity \(\lambda > 0\). Define
\[
X_t = \hat{X}_{N_t}, \quad t \geq 0.
\]
Then \((X_t)_{t \geq 0}\) is a continuous time Markov chain with generator
\[
A = \lambda(Q - I).
\]
Recall: \((X_t)_{t \geq 0}\) is a continuous time Markov chain starting from \(X_0 = x\).

\[ J_1 = \inf \{ t : X_t \neq x \}, \quad Y_1 = X_{J_1}. \]

**Theorem**

For \(x \neq y\), we have

\[ \mathbb{P}_x[J_1 > t, X_{J_1} = y] = e^{-q_x t} \frac{q_{xy}}{q_x}. \]

*In particular,*

- \( \mathbb{P}_x[J_1 > t] = e^{-q_x t} \)
- \( \mathbb{P}_x[X_{J_1} = y] = \frac{q_{xy}}{q_x} \)
- \( J_1 \) and \( X_{J_1} \) are independent.

**Remark:** if \( q_x = 0 \), we say that \( x \) is absorbing.
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