Recall: \((X_t)_{t \geq 0}\) is a continuous time Markov chain on countable state space with the following requirements:

- (Homogeneity) \(P[X_{t+s} = y \mid X_s = x] = P_t(x, y)\)
- (Right-continuity for the chain) For any \(t \geq 0\), there exists \(\epsilon > 0\), such that \(X_{t+s} = X_t\) for all \(s \in [0, \epsilon]\)
- (Non explosion) The explosion time \(\xi = \infty\)
- (Right-continuity in the semigroup) \(P_\epsilon \to P_0 = I\) as \(\epsilon \to 0\), pointwise for each entry.

Consider the transition semigroup \((P_t)_{t \geq 0}\)

- the infinitesimal generator \(A = \lim_{\epsilon \to 0}(P_\epsilon - I)\). We write \(A = P'_0\).
- Since \(P_{t+s} = P_tP_s\), we have \(P'_t = AP_t\)

Today’s goal:

- the infinitesimal generator \(A\) characterizes the chain
- irreducible, recurrent
Infinitesimal generator characterizes the transition semigroup

**Theorem**

Let \((X_t)_{t \geq 0}\) be a continuous time Markov chain with generator \(A\). Then the semigroup \((P_t)_{t \geq 0}\) is the minimal nonnegative solution to the backward equation

\[ P'_t = AP_t, \quad P_0 = I. \]

Recall

- the limits \(q_x = \lim_{\epsilon \to 0} (1 - P_\epsilon(x, x))/\epsilon, q_{xy} = \lim_{\epsilon \to 0} P_\epsilon(x, y)/\epsilon\) exist.
- the holding time \(J_1 : P_x[J_1 > t] = e^{-q_xt}\)
- the jump process : \(P_x[X_{J_1} = y] = q_{xy}/q_x\)
- \(J_1\) and \(X_{J_1}\) are independent
Irreducible

Suppose that $X = (X_t)_{t \geq 0}$ is a continuous time Markov chain.
- the jump times $J_0, J_1, J_2, \ldots : J_0 = 0, J_{n+1} = \inf\{t > J_n : X_t \neq X_{J_n}\}$.
- the jump chain $Y_0, Y_1, Y_2, \ldots : Y_n = X_{J_n}$
- the limits $q_x = \lim_{\epsilon \to 0} (1 - P_{\epsilon}(x, x))/\epsilon, q_{xy} = \lim_{\epsilon \to 0} P_{\epsilon}(x, y)/\epsilon$ exist.
- the holding time $S_x : \text{exponential with parameter } q_x$
- the jump process : $\mathbb{P}_x[X_{J_1} = y] = q_{xy}/q_x$

Definition

A continuous time Markov chain is irreducible if and only if its jump chain is irreducible.

Lemma

For $x, y \in \Omega$, the following statements are equivalent
- $\exists n \geq 1$ such that $\mathbb{P}_x[Y_n = y] > 0$.
- $\exists x_0 = x, x_1, \ldots, x_n = y$ such that $q_{x_0x_1} q_{x_1x_2} \cdots q_{x_{n-1}x_n} > 0$.
- $P_t(x, y) > 0$ for all $t > 0$
Recurrence

Suppose that $X$ is a continuous time Markov chain and that $Y$ is its jump chain.

**Definition**

A state $x$ is recurrent if $\mathbb{P}_x\left\{\{t : X_t = x\}\text{ is unbounded}\right\} = 1$.

A state $x$ is transient if $\mathbb{P}_x\left\{\{t : X_t = x\}\text{ is unbounded}\right\} < 1$.

**Theorem**

Let $X$ be an irreducible continuous time Markov chain.

- If $x$ is recurrent for $Y$, then $x$ is recurrent for $X$.
- If $x$ is transient for $Y$, then $x$ is transient for $X$.
- Either all states are recurrent, or all states are transient.

**Remark**

A state $x$ is recurrent for $X$ if and only if $\int_0^\infty P_t(x, x)\,dt = \infty$.

A state $x$ is transient for $X$ if and only if $\int_0^\infty P_t(x, x)\,dt < \infty$. 
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