Recall
Suppose that $P$ is irreducible with stationary measure $\pi$.

$$d(n) = \max_x \|P^n(x, \cdot) - \pi\|_{TV}, \quad t_{mix} = \min\{n : d(n) \leq 1/4\}.$$

Today’s Goal Use random times to give upper bound of $t_{mix}$

- Top-to-Random shuffle
- Stopping time and randomized stopping time
- Stationary time and strong stationary time
Top-to-Random shuffle

Consider the following method of shuffling a deck of $N$ cards:
Take the top card and insert it uniformly at random in the deck.

The successive arrangements of the deck are a random walk $(X_n)_{n \geq 0}$ on the group $S_N : N!$ possible permutations of the $N$ cards starting from $X_0 = (123 \cdots N)$.
The uniform measure is the stationary measure.

**Question** : How long must we shuffle until the orders in the deck is uniform?
A simpler question : How long must we shuffle until the original bottom card become uniform in the deck?

**Answer** : Let $\tau_{top}$ be the time one move after the first occasion when the original bottom card has moved to the top of the deck. The arrangements of cards at time $\tau_{top}$ is uniform in $S_N$. 
Top-to-Random shuffle

Theorem

Let \((X_n)_{n \geq 0}\) be the random walk on \(S_N\) corresponding to the top-to-random shuffle. Given at time \(n\) there are \(k\) cards under the original bottom card, each of the \(k!\) possibilities are equally likely. Therefore, \(X_{\tau_{\text{top}}}\) is uniform in \(S_N\).

Remark: The random time \(\tau_{\text{top}}\) is interesting, since \(X_{\tau_{\text{top}}}\) has exactly the stationary measure.
**Definition**

Given a sequence $(X_n)_{n \geq 0}$ of random variables, a number $\tau$, taking values in \{0, 1, 2, ..., $\infty$\}, is a stopping time for $(X_n)_{n \geq 0}$, if for each $n \geq 0$, the event $[\tau = n]$ is measurable with respect to $(X_0, X_1, ..., X_n)$; or equivalently, the indicator function $1_{[\tau=n]}$ is a function of the vector $(X_0, X_1, ..., X_n)$.

**Example** Fix a subset $A \subset \Omega$, define $\tau_A$ to be the first time that $(X_n)_{n \geq 0}$ hits $A$:

$$\tau_A = \min\{n : X_n \in A\}.$$

Then $\tau_A$ is stopping time. (Recall that $\tau_x$ and $\tau^+_x$ are stopping times.)
Lemma

Let $\tau$ be a random time, then the following four conditions are equivalent.

- $[\tau = n]$ is measurable w.r.t. $(X_0, X_1, \ldots, X_n)$
- $[\tau \leq n]$ is measurable w.r.t. $(X_0, X_1, \ldots, X_n)$
- $[\tau > n]$ is measurable w.r.t. $(X_0, X_1, \ldots, X_n)$
- $[\tau \geq n]$ is measurable w.r.t. $(X_0, X_1, \ldots, X_{n-1})$

Lemma

If $\tau$ and $\tau'$ are stopping times, then $\tau + \tau'$, $\tau \wedge \tau'$, and $\tau \vee \tau'$ are also stopping times.
Random mapping representation

Definition
A random mapping representation of a transition matrix $P$ on state space $\Omega$ is a function $f : \Omega \times \Lambda \rightarrow \Omega$, along with a $\Lambda$-valued random variable $Z$, satisfying

$$\mathbb{P}[f(x, Z) = y] = P(x, y).$$

**Question**: How is it related to Markov chain?
Let $(Z_n)_{n \geq 1}$ be i.i.d. with common law the same as $Z$. Let $X_0 \sim \mu$. Define $X_n = f(X_{n-1}, Z_n)$ for $n \geq 1$. Then $(X_n)_{n \geq 0}$ is a Markov chain with initial distribution $\mu$.

**Example**: Simple random walk on $N$-cycle. Set $\Lambda = \{-1, +1\}$, let $(Z_n)_{n \geq 1}$ be i.i.d. Bernoulli(1/2). Set

$$f(x, z) \equiv x + z \mod N.$$
Definition
A random mapping representation of a transition matrix $P$ on state space $\Omega$ is a function $f : \Omega \times \Lambda \to \Omega$, along with a $\Lambda$–valued random variable $Z$, satisfying

$$\mathbb{P}[f(x, Z) = y] = P(x, y).$$

Theorem
Every transition matrix on a finite state space has a random mapping representation.
Randomized stopping times

Suppose that the transition matrix $P$ has a random mapping representation $f : \Omega \times \Lambda \rightarrow \Omega$, along with a random variable $Z$, such that

$$
\mathbb{P}[f(x, Z) = y] = P(x, y).
$$

Let $(Z_n)_{n \geq 1}$ be a sequence of i.i.d. with the same law as $Z$. Define $X_n = f(X_{n-1}, Z_n)$ for $n \geq 1$. Then $(X_n)_{n \geq 0}$ is a Markov chain with transition matrix $P$.

Definition

A random time $\tau$ is called a randomized stopping time if it is a stopping time for the sequence $(Z_n)_{n \geq 1}$.

Remark The sequence $(Z_n)_n$ contains more information than the sequence $(X_n)_n$, therefore the stopping times for $(X_n)_n$ are randomized stopping times, but the reverse does not hold generally.
 Definition

Let $(X_n)_n$ be an irreducible Markov chain with stationary measure $\pi$. A stationary time $\tau$ for $(X_n)_n$ is a randomized stopping time such that $X_\tau \overset{d}{\sim} \pi$:

$$\mathbb{P}[X_\tau = x] = \pi(x), \quad \forall x.$$

A strong stationary time $\tau$ for $(X_n)_n$ is a randomized stopping time such that $X_\tau \overset{d}{\sim} \pi$ and $X_\tau \perp \tau$:

$$\mathbb{P}[X_\tau = x, \tau = n] = \pi(x)\mathbb{P}[\tau = n], \quad \forall x, n.$$

Example For the top-to-random shuffle, the time $\tau_{top}$ is strong stationary.
Example  Let \((X_n)_n\) be an irreducible Markov chain with state space \(\Omega\), stationary measure \(\pi\), and \(X_0 = x\). Let \(\xi\) be a \(\Omega\)-valued random variable with distribution \(\pi\) and it is independent of \((X_n)_n\). Define

\[
\tau = \min\{n \geq 0 : X_n = \xi\}.
\]

Then

- \(\tau\) is not a stopping time
- \(\tau\) is a randomized stopping time
- \(\tau\) is stationary
- \(\tau\) is not strong stationary
Strong stationary time

Theorem
Let \((X_n)_{n \geq 0}\) be an irreducible Markov chain with stationary measure \(\pi\). If \(\tau\) is a strong stationary time for \((X_n)\), then

\[
d(n) := \max_x ||P^n(x, \cdot) - \pi||_{TV} \leq \max_x \mathbb{P}_x[\tau > n].
\]

Lemma
For all \(n \geq 0\), \(\mathbb{P}[\tau \leq n, X_n = y] = \mathbb{P}[\tau \leq n] \pi(y)\).

Lemma
Define the separation distance \(S_x(n) = \max_y (1 - P^n(x, y)/\pi(y))\). Then \(S_x(n) \leq \mathbb{P}_x[\tau > n]\).

Lemma
\[
||P^n(x, \cdot) - \pi||_{TV} \leq S_x(n).
\]
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