Recall Suppose that $P$ is irreducible with stationary measure $\pi$.

$$d(n) = \max_x \| P^n(x, \cdot) - \pi \|_{TV}, \quad t_{mix} = \min\{n : d(n) \leq 1/4\}.$$

Today’s Goal Summary of the results on the mixing times.
- Upper bounds and lower bounds on mixing times
- Gambler’s ruin, Coupon collecting
- Random walk on hypercube
- Random walk on $N$–cycle
- Top-to-random shuffle
Upper bounds

Suppose that $P$ is irreducible with stationary distribution $\pi$.

**Theorem (Coupling of two Markov chains)**

Let $(X_n, Y_n)_{n \geq 0}$ be a coupling of Markov chains with transition matrix $P$ for which $X_0 = x$, $Y_0 = y$. Define $\tau$ to be their first meet time:

$$\tau = \min\{n \geq 0 : X_n = Y_n\}.$$  
Then

$$||P^n(x, \cdot) - P^n(y, \cdot)||_{TV} \leq \mathbb{P}_{x,y}[\tau > n]; \quad d(n) \leq \max_{x,y} \mathbb{P}_{x,y}[\tau > n].$$

**Theorem (Strong stationary time)**

Let $(X_n)_{n \geq 0}$ be a Markov chain with transition matrix $P$. If $\tau$ is a strong stationary time for $(X_n)$, then

$$d(n) := \max_x ||P^n(x, \cdot) - \pi||_{TV} \leq \max_x \mathbb{P}[\tau > n].$$
Lower bounds

Suppose that $P$ is irreducible with stationary measure $\pi$.

Theorem (Bottleneck ratio)

Define $Q(A, B) = \sum_{x \in A, y \in B} \pi(x) P(x, y)$, $\Phi(S) = Q(S, S^c) / \pi(S)$. The bottleneck ratio of the chain is defined to be

$$\Phi_* = \min\{\Phi(S) : \pi(S) \leq 1/2\}.$$

Then

$$t_{mix} \geq \frac{1}{4\Phi_*}.$$

Theorem (Distinguishing statistic)

Let $\mu$ and $\nu$ be two probability distributions on $\Omega$. Let $f$ be a real-valued function on $\Omega$. If

$$|\mu f - \nu f| \geq r\sigma,$$

where $\sigma^2 = \frac{1}{2}(\text{var}_\mu(f) + \text{var}_\nu(f))$, then

$$\|\mu - \nu\|_{TV} \geq \frac{r^2}{4 + r^2}.$$
Gambler’s ruin

Consider a gambler betting on the outcome of a sequence of independent fair coin tosses. If head, he gains one dollar. If tail, he loses one dollar. If he reaches a fortune of \( N \) dollars, he stops. If his purse is ever empty, he stops.

The gambler’s situation can be modeled by a Markov chain on the state space \( \{0, 1, \ldots, N\} \):
- \( X_0 \): initial money in purse
- \( X_n \): the gambler’s fortune at time \( n \)
- \( \tau \): the time that the gambler stops.

**Theorem**

Assume that \( X_0 = k \) for some \( 0 \leq k \leq N \). Then

\[
\mathbb{P}[X_\tau = N] = \frac{k}{N}, \quad \mathbb{E}[\tau] = k(N - k).
\]
Coupon collecting

A company issues \( N \) different types of coupons. A collector desires a complete set. The collector’s situation can be modeled by a Markov chain on the state space \( \{0, 1, \ldots, N\} \):

- \( X_0 = 0 \)
- \( X_n \): the number of different types among the collector’s first \( n \) coupons.
- \( \mathbb{P}[X_{n+1} = k + 1 \mid X_n = k] = (N - k)/N \),
- \( \mathbb{P}[X_{n+1} = k \mid X_n = k] = k/N \).
- \( \tau \): the first time that the collector obtains all \( N \) types.

Theorem

\[
\mathbb{E}[\tau] = N \sum_{k=1}^{N} \frac{1}{k} \approx N \log N.
\]

For any \( \alpha > 0 \), we have that

\[
\mathbb{P}[\tau > N \log N + \alpha N] \leq e^{-\alpha}.
\]
Random walk on hypercube

The lazy walk on hypercube can be constructed using the following random mapping representation: Uniformly select an element \((j, B)\) in \(\{1, \ldots, N\} \times \{0, 1\}\), and then update the coordinate \(j\) with \(B\).

Let \((Z_n = (j_n, B_n))_{n \geq 1}\) be i.i.d. \(\sim (j, B)\). At each step, the coordinate \(j_n\) of \(X_{n-1}\) is updated by \(B_n\). Define

\[
\tau = \min\{n : \{j_1, \ldots, j_n\} = \{1, \ldots, N\}\}.
\]

This is the first time that all the coordinates have been selected at least once for updating.

**Theorem**

There exists constants \(c > 0\), \(C < \infty\) such that

\[
CN \log N \geq t_{mix} \geq cN \log N.
\]

**Proof** Upper bound: strong stationary time.
Lower bound: distinguishing statistic.
**Random walk on \( N \)-cycle**

**Lazy walk**: it remains in current position with probability 1/2, moves left with probability 1/4, right with probability 1/4.

- It is irreducible.
- The stationary measure is the uniform measure.

**Theorem**

*For the lazy walk on \( N \)-cycle, there exists some constant \( c_0 > 0 \) such that*

\[
c_0 N^2 \leq t_{\text{mix}} \leq N^2.
\]

**Proof**

Upper bound: Coupling of two Markov chains.
Lower bound.
Top-to-random shuffle

Consider the following method of shuffling a deck of $N$ cards:
Take the top card and insert it uniformly at random in the deck.
The successive arrangements of the deck are a random walk $(X_n)_{n \geq 0}$
on the group $S_N$ starting from $X_0 = (123 \cdots N)$.
The uniform measure is the stationary measure.

Let $\tau_{top}$ be the time one move after the first occasion when the original bottom card has moved to the top of the deck. The arrangements of cards at time $\tau_{top}$ is uniform in $S_N$.

Theorem

There exist constant $c_0 \in (0, \infty)$ such that

$$N \log N - c_0 N \leq t_{mix} \leq N \log N + c_0 N.$$ 

Proof

Upper bound: $\tau_{top}$ is strong stationary.
Lower bound.
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