Recall: Reversible Markov chain: there exists a probability measure $\pi$ such that

$$\pi(x)P(x, y) = \pi(y)P(y, x), \quad \forall x, y \in \Omega.$$ 

- $\pi$ is stationary
- $\mathbb{P}_\pi[X_0 = x_0, \ldots, X_n = x_n] = \mathbb{P}_\pi[X_0 = x_n, \ldots, X_n = x_0].$

Today’s Goal: Electrical networks

- network, conductance, resistance
- voltage, current flow
- effective resistance
A network is a finite undirected connected graph $G = (V, E)$ endowed with non-negative numbers $\{c(e) : e \in E\}$.

- $c(e)$: conductance. Write $c(x, y)$ for $c(e)$ where $e = \{x, y\}$. Clearly $c(x, y) = c(y, x)$.
- $r(e) = 1 / c(e)$: resistance.
Weighted random walk on network

Definition
Consider the Markov chain on $V$ with transition matrix

$$P(x, y) = \frac{c(x, y)}{c(x)}, \quad \text{where } c(x) = \sum_y c(x, y).$$

This process is called the weighted random walk on $G$ with edge weights $\{c(e) : e \in E\}$.

This Markov chain is reversible with respect to the probability measure $\pi$ defined by

$$\pi(x) = \frac{c(x)}{c_G}, \quad \text{where } c_G = \sum_x c(x).$$

Therefore $\pi$ is stationary for $P$. 
Harmonic functions

\( \Omega \): state space; \( P \): the transition matrix, irreducible.

A function \( h : \Omega \to \mathbb{R} \) is harmonic at \( x \) if \( h(x) = \sum_y P(x, y)h(y) \).

Fix \( B \subset \Omega \), define the hitting time by

\[
\tau_B = \min\{n \geq 0 : X_n \in B\}.
\]

**Theorem**

Let \((X_n)_{n \geq 0}\) be a Markov chain with irreducible transition matrix \( P \). Let \( h_B : B \to \mathbb{R} \) be a function defined on \( B \). The function \( h : \Omega \to \mathbb{R} \) defined by

\[
h(x) = \mathbb{E}_x[h_B(X_{\tau_B})]
\]

is the unique extension of \( h_B \) such that

\[
h(x) = h_B(x), \quad \forall x \in B
\]

and that \( h \) is harmonic at all \( x \in \Omega \setminus B \).
Voltage

Definition
Consider a network \((G = (V, E), \{c(e) : e \in E\})\). We distinguish two vertices \(a\) (the source) and \(z\) (the sink). A voltage is a function on \(V\) which is harmonic on \(V \setminus \{a, z\}\).

Remark A voltage is completely determined by its boundary values \(W(a)\) and \(W(z)\).
Definition

Consider a function $\theta$ defined on oriented edges. The divergence of $\theta$ is defined by

$$\text{div}\theta(x) = \sum_{y : y \sim x} \theta(xy).$$

Definition

A flow from $a$ to $z$ is a function $\theta$ defined on oriented edges satisfying

1. $\theta$ is antisymmetric: $\theta(xy) = -\theta(yx)$;
2. $\text{div}\theta(x) = 0$ for all $x \in V \setminus \{a, z\}$ (Node Law);
3. $\text{div}\theta(a) \geq 0$.

We define the strength of a flow $\theta$ from $a$ to $z$ to be $||\theta|| = \text{div}\theta(a)$. A unit flow is a flow with strength 1.
Current flow

Definition
Given a voltage $W$ on the network, the current flow $I$ associated with $W$ is defined by

$$I(x\hat{y}) = \frac{W(x) - W(y)}{r(x, y)} = c(x, y)(W(x) - W(y)).$$

The current flow satisfies
- Ohm’s Law: $r(x, y)I(x\hat{y}) = W(x) - W(y);$  
- Cycle Law: if the oriented edges $\vec{e}_1, \ldots, \vec{e}_m$ form an oriented cycle, then

$$\sum_{j=1}^{m} r(\vec{e}_j)I(\vec{e}_j) = 0.$$

Theorem
If $\theta$ is a flow from $a$ to $z$ satisfying Cycle Law for any cycle and $||\theta|| = ||I||$, then $\theta = I.$
Effective resistance

Definition
Given a network, suppose that $W$ is a voltage and $I$ is the corresponding current flow. Define the effective resistance between $a$ and $z$ by

$$R(a \leftrightarrow z) = \frac{W(a) - W(z)}{||I||}.$$ 

Theorem (Effective resistance and Escape probability)
For any $a, z \in \Omega$, consider the weighted random walk on the network, we have

$$P_a[\tau_z < \tau_a^+] = \frac{1}{c(a)R(a \leftrightarrow z)}.$$
Effective resistance

Definition

The Green's function for a random walk stopped at a stopping time $\tau$ is defined by

$$G_\tau(a, x) = \mathbb{E}_a[\# \text{visits to } x \text{ before } \tau] = \sum_{n \geq 0} \mathbb{P}_a[X_n = x, n < \tau].$$

Theorem (Effective resistance and Green's function)

$$G_{\tau_z}(a, a) = c(a)R(a \leftrightarrow z).$$