Fall 2012 18.440 Final Exam: 100 points
Carefully and clearly show your work on each problem (without writing anything that is technically not true) and put a box around each of your final computations.

1. (10 points) Lisa’s truck has three states: broken (in Lisa’s possession), working (in Lisa’s possession), and in the shop. Denote these states B, W, and S.
   (i) Each morning the truck starts out B, it has a 1/2 chance of staying B and a 1/2 chance of switching to S by the next morning.
   (ii) Each morning the truck starts out W, it has 9/10 chance of staying W, and a 1/10 chance of switching to B by the next morning.
   (iii) Each morning the truck starts out S, it has a 1/2 chance of staying S and a 1/2 chance of switching to W by the next morning.

   Answer the following:
   (a) Write the three-by-three Markov transition matrix for this problem.  
       ANSWER: Ordering the states B, W, S, we may write the Markov chain matrix as 
       \[ M = \begin{pmatrix} .5 & 0 & .5 \\ .1 & .9 & 0 \\ 0 & .5 & .5 \end{pmatrix}. \]
   (b) If the truck starts out W on one morning, what is the probability that it will start out B two days later?  
       ANSWER: 
       \[ (9/10)(1/10) + (1/10)(1/2) = .09 + .05 = .14 \]
   (c) Over the long term, what fraction of mornings does the truck start out in each of the three states, B, S, and W?  
       ANSWER: We find the stationarity probability vector \( \pi = (\pi_B, \pi_W, \pi_S) = (1/7, 5/7, 1/7) \) by solving \( \pi M = \pi \) (with components of \( \pi \) summing to 1).

2. (10 points) Suppose that \( X_1, X_2, X_3, \ldots \) is an infinite sequence of independent random variables which are each equal to 1 with probability 1/2 and \(-1\) with probability 1/2. Write \( Y_n = \sum_{i=1}^{n} X_i \). Answer the following:
   (a) What is the probability that \( Y_n \) reaches 10 before the first time that it reaches \(-30\)?  
       ANSWER: Probability \( p \) satisfies \( 10p + (-30)(1-p) = 0 \), so \( 40p = 30 \) and \( p = 3/4 \).
(b) In which of the cases below is the sequence $Z_n$ a martingale? (Just circle the corresponding letters.)

(i) $Z_n = X_n + Y_n$ **ANSWER: NO**

(ii) $Z_n = \prod_{i=1}^{n}(2X_i + 1)$ **ANSWER: YES**

(iii) $Z_n = \prod_{i=1}^{n}(-X_i + 1)$ **ANSWER: YES**

(iv) $Z_n = \sum_{i=1}^{n}Y_i$ **ANSWER: NO**

(v) $Z_n = \sum_{i=2}^{n}X_iX_{i-1}$ **ANSWER: YES**

3. (10 points) Ten people throw their hats into a box and then randomly redistribute the hats among themselves (each person getting one hat, all 10! permutations equally likely). Let $N$ be the number of people who get their own hats back. Compute the following:

(a) $E[N^2]$ **ANSWER:** Let $N_i$ be 1 if $i$th person gets own hat, zero otherwise. Then

$E[(\sum N_i)^2] = \sum_{i=1}^{10} \sum_{j=1}^{10} E[N_iN_j] = 90(1/90) + 10(1/10) = 2.$

(b) $P(N = 8)$ **ANSWER:** There are $\binom{10}{2}$ ways to pick a pair of people to have swapped hats. So answer is $\binom{10}{2}/10!$.

4. (10 points) When Harry’s cell phone is on, the times when he receives new text messages form a Poisson process with parameter $\lambda_T = 3$/minute. The times at which he receives new email messages form an independent Poisson process with parameter $\lambda_E = 1$/minute. He receives personal messages on Facebook as an independent Poisson process with rate $\lambda_F = 2$/minute.

(a) After catching up on existing messages one morning, Harry begins to wait for new messages to arrive. Let $X$ be the amount of time (in minutes) that Harry has to wait to receive his first text message. Write down the probability density function for $X$. **ANSWER:**

The time is exponential with parameter $\lambda_T = 3$, so density function is $f(x) = 3e^{-3x}$ for $x \geq 0$.

(b) Compute the probability that Harry receives 10 new messages total (including email, text, and Facebook) during his first two minutes of waiting. **ANSWER:** Number total in two minutes is Poisson with rate $\lambda = 2(\lambda_E + \lambda_T + \lambda_F) = 12$. So answer is $\lambda^k e^{-\lambda}/k! = 12^{10} e^{-12}/10!$. 

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(c) Let \( Y \) be the amount of time elapsed before the third email message. Compute \( \text{Var}(Y) \). **ANSWER:** Variance of time till email message is \( \frac{1}{\lambda^2} = 1 \). Memoryless property and additivity of variance of independent sums gives \( \text{Var}(S) = 3 \).

(d) What is the probability that Harry receives no messages of any kind during his first five minutes of waiting? **ANSWER:** Time till first message is exponential with parameter 6. Probability this time exceeds 5 is \( e^{-30} \).

5. (10 points) Suppose that \( X \) and \( Y \) have a joint density function \( f \) given by

\[
    f(x, y) = \begin{cases} 
        \frac{1}{\pi} & x^2 + y^2 < 1 \\
        0 & x^2 + y^2 \geq 1 
    \end{cases}
\]

(a) Compute the probability density function \( f_X \) for \( X \). **ANSWER:**

\[
    f_X(x) = \int_{-\infty}^{\infty} f(x, y) dy = \begin{cases} 
        \frac{1}{2} & -1 \leq x \leq 1 \\
        0 & \text{otherwise}
    \end{cases}
\]

(b) Express \( E[\sin(XY)] \) as a double integral. (You don’t have to explicitly evaluate the integral.) **ANSWER:**

\[
    \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{\pi} \sin(xy) dy dx.
\]

6. (10 points) Let \( X \) be the number on a standard die roll (i.e., each of \( \{1, 2, 3, 4, 5, 6\} \) is equally likely) and \( Y \) the number on an independent standard die roll. Write \( Z = X + Y \).

(a) Compute the conditional probability \( P[X = 6|Z = 8] \). **ANSWER:** 1/5

(b) Compute the conditional expectation \( E[Y|Z] \) as a function of \( Z \) (for \( Z \in \{2, 3, 4, \ldots, 12\} \)). **ANSWER:**

\[
    Z = E[Z|Z] = E[X + Y|Z] = E[X|Z] + E[Y|Z]. \quad \text{By symmetry,}
\]

\[
\]

7. (10 points) Suppose that \( X_i \) are i.i.d. random variables, each of which assumes a value in \( \{-1, 0, 1\} \), each with probability 1/3.

(a) Compute the moment generating function for \( X_1 \). **ANSWER:**

\[
    E e^{tX_1} = (e^{-t} + 1 + e^t)/3.
\]
(b) Compute the moment generating function for the sum $\sum_{i=1}^{n} X_i$.  
\textbf{ANSWER:} $(e^{-t} + 1 + e^t)^n / 3^n$

8. (10 points) Let $X$ and $Y$ be independent random variables. Suppose $X$ takes values in \{1, 2\} each with probability 1/2 and $Y$ takes values in \{1, 2, 3, 4\} each with probability 1/4. Write $Z = X + Y$.

(a) Compute the entropies $H(X)$ and $H(Y)$.  \textbf{ANSWER:} $\log 2 = 1$ and $\log 4 = 2$.

(b) Compute $H(X, Z)$.  \textbf{ANSWER:} $\log 2 + \log 4 = \log 8 = 3$.

(c) Compute $H(X + Y)$.  \textbf{ANSWER:}

$$
\sum_{i=1}^{6} P(X+Y = i)(-\log P(X+Y = i)) = 2 \cdot \frac{1}{8} \log 8 + 3 \cdot \frac{1}{4} \log 4 = 6/8 + 6/4 = 9/4.
$$

9. (10 points) Let $X$ be a normal random variable with mean 0 and variance 1.

(a) Compute $\mathbb{E}[e^X]$.  \textbf{ANSWER:}

$$
E(e^X) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} e^x dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-(x^2 - 2x + 1)/2 + 1/2} dx = e^{1/2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-(x-1)^2/2} dx = e^{1/2}.
$$

(b) Compute $\mathbb{E}[e^X 1_{X>0}]$.  \textbf{ANSWER:}

$$
E(e^X 1_{X>0}) = \int_{0}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} e^x dx \\
= \int_{0}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-(x^2 - 2x + 1)/2 + 1/2} dx \\
= e^{1/2} \int_{0}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-(x-1)^2/2} dx \\
= e^{1/2} \int_{-1}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx = e^{1/2}(1 - \Phi(-1)) = e^{1/2}\Phi(1),
$$

where $\Phi(a) = \int_{-\infty}^{a} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$.

(c) Compute $\mathbb{E}[X^2 + 2X - 5]$.  \textbf{ANSWER:}

$$
$$
10. (10 points) Let $X$ be uniformly distributed random variable on $[0, 1]$.

(a) Compute the variance of $X$. **ANSWER:** $E[X^2] = \int_0^1 x^2 \, dx = 1/3$
and $E[X] = 1/2$ so $\text{Var}[X] = E[X^2] - E[X]^2 = 1/12$.

(b) Compute the variance of $3X + 5$. **ANSWER:** $9\text{Var}[X] = 3/4$.

(c) If $X_1, \ldots, X_n$ are independent copies of $X$, and
$Z = \max\{X_1, X_2, \ldots, X_n\}$, then what is the cumulative distribution function $F_Z$? **ANSWER:**

$$F_Z(a) = P\{Z \leq a\} = \prod_{i=1}^n P\{X_i \leq a\} = F_{X_1}(a)^n = \begin{cases} 0 & a < 0 \\ a^n & a \in [0, 1] \\ 1 & a > 1 \end{cases}$$