Spring 2014 18.440 Final Exam Solutions

1. (10 points) Let $X$ be a uniformly distributed random variable on $[-1, 1]$.

(a) Compute the variance of $X^2$. Answer:

$$
\text{Var}(X^2) = E[(X^2)^2] - E[X^2]^2,
$$

and

$$
E[X^2] = \int_{-1}^{1} \frac{x^2}{2} \, dx = \frac{x^3}{6} \bigg|_{-1}^{1} = \frac{1}{3},
$$

$$
E[(X^2)^2] = E[X^4] = \int_{-1}^{1} \frac{x^4}{2} \, dx = \frac{x^5}{10} \bigg|_{-1}^{1} = \frac{1}{5},
$$

so $\text{Var}(X^2) = E[(X^2)^2] - E[X^2]^2 = \frac{1}{5} - \left(\frac{1}{3}\right)^2 = \frac{1}{5} - \frac{1}{9} = \frac{4}{45}.$

(b) If $X_1, \ldots, X_n$ are independent copies of $X$, and $Z = \max\{X_1, X_2, \ldots, X_n\}$, then what is the cumulative distribution function $F_Z$? Answer: $F_{X_1}(a) = (a + 1)/2$ for $a \in [-1, 1]$. Thus

$$
F_Z(a) = F_{X_1}(a)F_{X_2}(a)\ldots F_{X_n}(a) = \begin{cases} 
\frac{(a+1)^n}{2^n} & a \in [-1, 1] \\
0 & a < -1 \\
1 & a > 1
\end{cases}
$$

2. (10 points) A certain bench at a popular park can hold up to two people. People in this park walk in pairs or alone, but nobody ever sits down next to a stranger. They are just not friendly in that particular way. Individuals or pairs who sit on a bench stay for at least 1 minute, and tend to stay for 4 minutes on average. Transition probabilities are as follows:

(i) If the bench is empty, then by the next minute it has a 1/2 chance of being empty, a 1/4 chance of being occupied by 1 person, and a 1/4 chance of being occupied by 2 people.

(ii) If it has 1 person, then by the next minute it has 1/4 chance of being empty and a 3/4 chance of remaining occupied by 1 person.

(iii) If it has 2 people then by the next minute it has 1/4 chance of being empty and a 3/4 chance of remaining occupied by 2 people.
(a) Use E, S, D to denote respectively the states empty, singly occupied, and doubly occupied. Write the three-by-three Markov transition matrix for this problem, labeling columns and rows by E, S, and D. 

**ANSWER:**

$$
\begin{pmatrix}
1/2 & 1/4 & 1/4 \\
1/4 & 3/4 & 0 \\
1/4 & 0 & 3/4 \\
\end{pmatrix}
$$

(b) If the bench is empty, what is the probability it will be empty two minutes later? **ANSWER:** 1/2 + 1/4 + 1/4 = 6/16 = 3/8.

(c) Over the long term, what fraction of the time does the bench spend in each of the three states? **ANSWER:** We know

$$
(E \ S \ D) \begin{pmatrix}
1/2 & 1/4 & 1/4 \\
1/4 & 3/4 & 0 \\
1/4 & 0 & 3/4 \\
\end{pmatrix} = (E \ S \ D)
$$


3. (10 points) Eight people throw their hats into a box and then randomly redistribute the hats among themselves (each person getting one hat, all 8! permutations equally likely). Let $N$ be the number of people who get their own hats back. Compute the following:

(a) $E[N]$ **ANSWER:** 8 × 1/8 = 1

(b) $P(N = 7)$ **ANSWER:** 0 since if seven get their own hat, then the eighth must also.

(c) $P(N = 0)$ **ANSWER:** This is an inclusion exclusion problem. Let $A_i$ be the event that the $i$th person gets own hat. Then

$$
P(N > 0) = P(A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_8) = \sum \limits_i P(A_i) - \sum \limits_{i<j} P(A_i \cap A_j) + \sum \limits_{i<j<k} P(A_i \cap A_j \cap A_k) - \ldots
$$

$$
= \frac{8}{1} \cdot \frac{1}{8} - \frac{8}{2} \cdot \frac{1}{8} \cdot \frac{1}{7} + \frac{8}{3} \cdot \frac{1}{8} \cdot \frac{1}{7} \cdot \frac{1}{6} - \ldots
$$

$$
= 1/1! - 1/2! + 1/3! + \ldots - 1/8!
$$

Thus,

$$
P(N = 0) = 1 - P(N > 0) = 1-1/1!+1/2!-1/3!+1/4!+1/5!-1/6!+1/7!-1/8! \approx 1/e.
$$
4. (10 points) Suppose that $X_1, X_2, X_3, \ldots$ is an infinite sequence of independent random variables which are each equal to 5 with probability 1/2 and −5 with probability 1/2. Write $Y_n = \sum_{i=1}^{n} X_i$. Answer the following:

(a) What is the probability that $Y_n$ reaches 65 before the first time that it reaches −15? \textbf{ANSWER:} $Y_n$ is a martingale, so by the optional stopping theorem, we have $E[Y_T] = Y_0 = 1$ (where $T = \min\{n : Y_n \in \{-15, 65\}\}$). We thus find

$$0 = Y_0 = E[Y_T] = 65p + (-15)(1 - p)$$

so $80p = 15$ and $p = 3/16$.

(b) In which of the cases below is the sequence $Z_n$ a martingale? (Just circle the corresponding letters.)

(i) $Z_n = 5X_n$
(ii) $Z_n = 5^{-n} \prod_{i=1}^{n} X_i$
(iii) $Z_n = \prod_{i=1}^{n} X_i^2$
(iv) $Z_n = 17$
(v) $Z_n = X_n - 4$

\textbf{ANSWER:} (iv) only.

5. (10 points) Suppose that $X$ and $Y$ are independent exponential random variables with parameter $\lambda = 2$. Write $Z = \min\{X, Y\}$

(a) Compute the probability density function for $Z$. \textbf{ANSWER:} $Z$ is exponential with parameter $\lambda + \lambda = 4$ so $F_Z(t) = 4e^{-4t}$ for $t \geq 0$.

(b) Express $E[\cos(X^2Y^3)]$ as a double integral. (You don’t have to explicitly evaluate the integral.) \textbf{ANSWER:}

$$\int_0^\infty \int_0^\infty \cos(x^2y^3) \cdot 2e^{-2x} \cdot 2e^{-2y}dydx$$

6. (10 points) Let $X_1, X_2, X_3$ be independent standard die rolls (i.e., each of \{1, 2, 3, 4, 5, 6\} is equally likely). Write $Z = X_1 + X_2 + X_3$.

(a) Compute the conditional probability $P[X_1 = 6|Z = 16]$. \textbf{ANSWER:}

One can enumerate the six possibilities that add up to 16. These are

\{(4, 6, 6), (6, 4, 6), (6, 6, 4), (6, 5, 5), (5, 6, 5), (5, 5, 6)\}.

Of these, three have $X_1 = 6$, so $P[X_1 = 6|Z = 16] = 1/2$.

(b) Compute the conditional expectation $E[X_2|Z]$ as a function of $Z$ (for $Z \in \{3, 4, 5, \ldots, 18\}$). \textbf{ANSWER:} Note that

$E[X_1 + X_2 + X_3|Z] = E[Z|Z] = Z$. So by symmetry and additivity of conditional expectation we find $E[X_2|Z] = Z/3$. 

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7. (10 points) Suppose that $X_i$ are i.i.d. uniform random variables on $[0, 1]$.

(a) Compute the moment generating function for $X_1$. **ANSWER:** 

$$E(e^{tX_1}) = \int_0^1 e^{tx} dx = \frac{e^t - 1}{t}.$$

(b) Compute the moment generating function for the sum $\sum_{i=1}^n X_i$. **ANSWER:** $(\frac{e^t - 1}{t})^n$

8. (10 points) Let $X$ be a normal random variable with mean 0 and variance 5.

(a) Compute $E[e^X]$. **ANSWER:** 

$$E[e^{tX}] = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/(2\cdot5)} e^{tx} dx.$$ 

A complete the square trick allows one to evaluate this and obtain $e^{t/2}$.

(b) Compute $E[X^9 + X^3 - 50X + 7]$. **ANSWER:** 

$$E[X^9] = E[X^7] = E[X] = 0 \text{ by symmetry, so}$$ 


9. (10 points) Let $X$ and $Y$ be independent random variables. Suppose $X$ takes values $\{1, 2\}$ each with probability $1/2$ and $Y$ takes values $\{1, 2, 3\}$ each with probability $1/3$. Write $Z = X + Y$.

(a) Compute the entropies $H(X)$ and $H(Y)$. **ANSWER:** 

$$H(X) = -(1/2) \log \frac{1}{2} - (1/2) \log \frac{1}{2} = - \log \frac{1}{2} = \log 2.$$ 

Similarly, 

$$H(Y) = -(1/3) \log \frac{1}{3} - (1/3) \log \frac{1}{3} - (1/3) \log \frac{1}{3} = - \log \frac{1}{3} = \log 3.$$

(b) Compute $H(X, Z)$. **ANSWER:** 

$$H(X, Z) = H(X, Y) = H(X) + H(Y) = \log 6.$$ 

(c) Compute $H(2^X 3^Y)$. **ANSWER:** Also $\log 6$, since each distinct $(X, Y)$ pair gives a distinct number for $2^X 3^Y$.

10. (10 points) Suppose that $X_1, X_2, X_3, \ldots$ is an infinite sequence of independent random variables which are each equal to 2 with probability 1/3 and .5 with probability 2/3. Let $Y_0 = 1$ and $Y_n = \prod_{i=1}^n X_i$ for $n \geq 1$.

(a) What is the probability that $Y_n$ reaches 4 before the first time that it reaches $\frac{1}{64}$? **ANSWER:** $Y_n$ is a martingale, so by the optional stopping theorem, $E[Y_T] = Y_0 = 1$ (where 

$$T = \min\{n : Y_n \in \{1/64, 4\}\}$$). Thus 


Solving yields $p = 63/255 = 21/85$. 


(b) Find the mean and variance of $\log Y_{400}$. \textbf{ANSWER:} $\log X_1$ is log 2 with probability $1/3$ and $-\log 2$ with probability $2/3$. So

$$E[\log X_1] = \frac{1}{3} \log 2 + \frac{2}{3} (-\log 2) = -\frac{\log 2}{3}.$$ 

Similarly,

$$E[(\log X_1)^2] = \frac{1}{3} (\log 2)^2 + \frac{2}{3} (-\log 2)^2 = (\log 2)^2.$$ 

Thus,

$$\text{Var}(X_1) = E[(\log X_1)^2] - E[\log X_1]^2 = (\log 2)^2 - \left(-\frac{\log 2}{3}\right)^2 = (\log 2)^2 \left(1 - \frac{1}{9}\right) = \frac{8}{9} (\log 2)^2.$$ 

Multiplying, we find $E[\log Y_{400}] = 400E[\log X_1] = -400(\log 2)/3$.

And $\text{Var}[\log Y_{400}] = (3200/9)(\log 2)^2$.

(c) Compute $EY_{100}$. \textbf{ANSWER:} Since $Y_n$ is a martingale, we have $E[Y_{100}] = 1$. This can also be derived by noting that for independent random variables, the expectation of a product is the product of the expectations.