18.440 Midterm 1 Solutions, Fall 2011: 50 minutes, 100 points

1. (20 points) Jill goes fishing. During each minute she fishes, there is a 1/600 chance that she catches a fish (independently of all other minutes). Assume that she fishes for 15 hours (900 minutes). Let \( N \) be the total number of fish she catches.

(a) Compute \( E[N] \) and \( \text{Var}[N] \). (Give exact answers, not approximate ones.) \textbf{ANSWER:} By additivity of expectation \( E[N] = 900/600 = 3/2 \). By variance additivity for independent random variables \( \text{Var}[N] = 900(1/600)(599/600) \)

(b) Compute the probability she catches exactly 3 fish. Give an \textit{exact answer}. \textbf{ANSWER:} \((900)_3(1/600)^3(599/600)^{897}\)

(c) Now use a Poisson random variable calculation to \textit{approximate} the probability that she catches exactly 3 fish. \textbf{ANSWER:} \( N \) is approximately Poisson with \( \lambda = 900/600 = 3/2 \). So \( P\{N = 3\} \approx e^{-\lambda} \lambda^3/3! = e^{-3/2} \frac{9}{16} \).

2. (10 points) Given ten people in a room, what is the probability that no two were born in the same month? (Assume that all of the 12^{10} ways of assigning birthday months to the ten people are equally likely.) \textbf{ANSWER:} \( \frac{(12)_{10!}}{12^{10}} \)

3. (10 points) Suppose that \( X \), \( Y \) and \( Z \) are independent random variables such that each is equal to 0 with probability .5 and 1 with probability .5.

(a) Compute the conditional probability \( P\{X + Y + Z = 1 | X - Y = 0\} \). \textbf{ANSWER:} \( Both \ events \ occur \ if \ and \ only \ if \ both \ X = Y = 0 \ and \ Z = 1. \ So \ P\{X + Y + Z = 1, X - Y = 0\} = 1/8 \ and \ P\{X - Y = 0\} = 1/2. \ Thus \ P\{X + Y + Z = 1 | X - Y = 0\} = (1/8)/(1/2) = 1/4. \)

(b) Are the events \( \{X = Y\} \) and \( \{Y = Z\} \) and \( \{X = Z\} \) independent? Are they pairwise independent? Explain. \textbf{ANSWER:} Not independent. Each event has probability 1/2 but probability all events occur is 1/4 \( \neq (1/2)^3 \). Are pairwise independent, since probability of any two occurring is \( (1/2)^2 = 1/4 \).

4. (20 points) Consider an infinite sequence of independent tosses of a coin that comes up heads with probability \( p \).
(a) Let $X$ be such that the first heads appears on the $X$th toss. In other words, $X$ is the number of tosses required to obtain a heads. Compute (in terms of $p$) the expectation and variance of $X$. 

**ANSWER:** Recall or derive: $E[X] = \sum_{k=1}^{\infty} q^{k-1} p k$, where $q = 1 - p$. Cute trick: write $E[X - 1] = \sum_{k=1}^{\infty} q^{k-1} p(k - 1)$. Setting $j = k - 1$, we have $E[X - 1] = q \sum_{j=0}^{\infty} q^{j} p j = qE[X]$. Thus $E[X] - 1 = qE[X]$ and solving for $E[X]$ gives $E[X] = 1/(1 - q) = 1/p$. Similarly, recall or derive: $E[X^2] = \sum_{k=1}^{\infty} q^{k-1}pk^2$. Cute trick: $E[(X - 1)^2] = \sum_{k=1}^{\infty} q^{k-1}p(k - 1)^2$. Setting $j = k - 1$, we have $E[(X - 1)^2] = q \sum_{j=0}^{\infty} q^{j} p j^2 = qE[X^2]$. Thus $E[(X - 1)^2] = E[X^2] - 2X + 1 = E[X^2] - 2/p + 1 = qE[X^2]$. Solving for $E[X^2]$ gives $(1 - q)E[X^2] = pE[X^2] = 2/p - 1$, so $E[X^2] = (2 - p)/p^2$ and $\text{Var}[X] = 1/p^2$.

(b) Let $Y$ be such that the fifth heads appears on the $Y$th toss. Compute (in terms of $p$) the expectation and variance of $Y$. **ANSWER:** By additivity of expectation and variance (for independent random variables) we obtain $E[Y] = 5/p$ and $\text{Var}[Y] = 5(1 - p)/p^2$.

5. (20 points) Suppose that $X$ is continuous random variable with probability density function $f_X(x) = \begin{cases} e^{-x} & x > 0 \\ 0 & x < 0 \end{cases}$. Compute the following:

(a) The expectation $E[X]$. **ANSWER:** $E[X] = \int_{-\infty}^{\infty} x f_X(x)dx = \int_0^{\infty} e^{-x} x dx = 1$.

(b) The probability $P\{X \in [-50, 50]\}$. **ANSWER:** $P\{X \in [-50, 50]\} = \int_{-50}^{50} f_X(x)dx = \int_0^{50} e^{-x} dx = 1 - e^{-50}$

(c) The cumulative distribution function $F_X$. **ANSWER:** $F_X(a) = \int_{-\infty}^{a} f_X(x)dx = \begin{cases} 0 & a \leq 0 \\ \int_0^{a} e^{-x} dx = 1 - e^{-a} & a > 0 \end{cases}$

6. (20 points) A group of 52 people (labeled 1, 2, 3, . . . , 52) toss their hats into a box, mix them up, and return one hat to each person (all 52! permutations equally likely). Compute the following:

(a) The probability that the first 26 hats all get their own hats. **ANSWER:** $\frac{1}{52} \cdot \frac{1}{51} \cdot \ldots \cdot \frac{1}{27} = \frac{26!}{52!}$
(b) The probability that there are 26 pairs of people whose hats are switched: i.e., each pair can be labeled \((a, b)\), such that \(a\) got \(b\)'s hat and \(b\) got \(a\)'s hat. **Answer:** Have \(\binom{52}{2, 2, \ldots, 2} = \frac{52!}{(2^{26})}\) ways to choose ordered list of 26 pairs. Dividing by 26! gives number of unordered collections of pairs. So we get \(\frac{\frac{52!}{2^{26}26!}}{52!}\) permutations of desired type. Dividing by 52! gives probability \(\frac{1}{2^{26}26!}\).