18.440 Midterm 2, Spring 2014: 50 minutes, 100 points

1. (20 points) Consider a sequence of independent tosses of a coin that is biased so that it comes up heads with probability 3/4 and tails with probability 1/4. Let $X_i$ be 1 if the $i$th toss comes up heads and 0 otherwise.

(a) Compute $E[X_1]$ and $\text{Var}[X_1]$. \textbf{ANSWER:} $E[X_1] = 3/4$ and $E[X_1^2] = 3/4$ so


(b) Compute $\text{Var}[X_1 + 2X_2 + 3X_3 + 4X_4]$. \textbf{ANSWER:} Using previous problem, additivity of variance for independent random variables, and general fact that $\text{Var}[aY] = a^2\text{Var}[Y]$, we find that

$$\text{Var}[X_1 + 2X_2 + 3X_3 + 4X_4] = (3/16)(1 + 4 + 9 + 16) = 90/16 = 45/8.$$ 

(c) Let $Y$ be the number of heads in the first 4800 tosses of the biased coin, i.e.,

$$Y = \sum_{i=1}^{4800} X_i.$$ 

Use a normal random variable to approximate the probability that $Y \geq 3690$. You may use the function $\Phi(a) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a} e^{-x^2/2} dx$ in your answer. \textbf{ANSWER:} $Y$ has expectation $4800E[X_1] = 3600$. It has variance $4800\text{Var}[X_1] = 900$ and standard deviation 30. We are looking for the probability that $Y$ is more than three standard deviations above its mean. This is approximately the probability that standard normal random variable is three standard deviations above its mean, which is $1 - \Phi(3)$. 

2. (10 points) Suppose that a fair six-sided die is rolled just once. Let $X \in \{1, 2, 3, 4, 5, 6\}$ be the number that comes up. Let $Y$ be 1 if the number on the die is in $\{1, 2, 3\}$ and 0 otherwise.

(a) What is the conditional expectation of $X$ given that $Y = 0$? \textbf{ANSWER:} Given that $Y$ is zero, $X$ is conditionally uniform on \{4, 5, 6\}, so the conditional expectation is 5.

(b) What is the conditional variance of $Y$ given that $X = 2$? \textbf{ANSWER:} Given that $X$ is 2, the conditional probability that $Y = 1$ is one, so the conditional variance is 0.
3. (20 points) Let $X$ be a uniform random variable on the set $\{-2, -1, 0, 1, 2\}$. That is, $X$ takes each of these values with probability $1/5$. Let $Y$ be an independent random variable with the same law as $X$, and write $Z = X + Y$.

(a) What is the moment generating function $M_X(t)$? **ANSWER:**

$$M_X(t) = E[e^{tX}] = \frac{1}{5}(e^{-2t} + e^{-t} + e^0 + e^t + e^{2t}).$$

(b) What is the moment generating function $M_Z(t)$? **ANSWER:**

$$M_Z(t) = M_X(t)M_Y(t) = \left[\frac{1}{5}(e^{-2t} + e^{-t} + e^0 + e^t + e^{2t})\right]^2.$$ 

4. (20 points) Two soccer teams, the Lions and the Tigers, begin an infinite soccer games starting at time zero. Suppose that the times at which the Lions score a goal form a Poisson point process with rate $\lambda_L = 2$/hour. Suppose that the times at which the Tigers score a goal form a Poisson point process with rate $\lambda_T = 3$/hour.

(a) Write down the probability density function for the amount of time until the first goal by the Lions. **ANSWER:** This is an exponential random variable with parameter $\lambda_L$. So the density function on $[0, \infty)$ is $f(x) = \lambda_L e^{-\lambda_L x} = 2e^{-2x}$.

(c) Write down the probability density function for the amount of time until the first goal by either team is scored. **ANSWER:** Recall that the minimum of two exponential random variables with parameters $\lambda_L$ and $\lambda_T$ is an exponential random variable with parameter $\lambda_L + \lambda_T = 5$. So the density function on $[0, \infty)$ is $f(x) = 5e^{-5x}$.

(c) Compute the probability that the Tigers score no goals at all during the first two hours. **ANSWER:** The probability that an exponential random of parameter $\lambda$ is at least $a$ is given by $e^{-\lambda a}$. Plugging in $\lambda = 3$ and $a = 2$ we get $e^{-6}$.

(d) Compute the probability that the Lions score exactly three goals during the first hour. **ANSWER:** The number of goals scored by the Lions during the first hour is a Poisson random variable with parameter $\lambda = \lambda_L = 2$. The probability that this is equal to a given $k$ is given by $e^{-\lambda} \lambda^k / k!$. Plugging in $k = 3$ and $\lambda = 2$ we get

$$e^{-2} * 3! = \frac{4}{3e^2}.$$
5. (20 points) Let \( X \) and \( Y \) be independent uniform random variables on \([0,1]\). Write \( Z = X + Y \). Write \( W = \max\{X,Y\} \).

(a) Compute and draw a graph of the probability density function \( f_Z \).

**ANSWER:** This is given by

\[
f_Z(x) = \begin{cases} 
0 & x \leq 0 \\
x & 0 < x \leq 1 \\
2 - x & 1 < x \leq 2 \\
0 & x \geq 2 
\end{cases}
\]

(b) Compute and draw a graph of the cumulative distribution function \( F_W \).

**ANSWER:** \( F_W(a) = \begin{cases} 
0 & a < 0 \\
a^2 & 0 \leq a \leq 1 \\
1 & a > 1 
\end{cases} \)

(c) Compute the variances \( \text{Var}(X) \), \( \text{Var}(Y) \), and \( \text{Var}(Z) \).

**ANSWER:**

\[ \text{Var}(X) = E[X^2] - (E[X])^2 = \int_0^1 x^2 \, dx - (1/2)^2 = 1/3 - 1/4 = 1/12. \]

Then \( \text{Var}(Y) = \text{Var}(X) \) and \( \text{Var}(Z) = \text{Var}(X) + \text{Var}(Y) = 2/12 \).

(d) Compute the covariance \( \text{Cov}(Y,Z) \) and the correlation coefficient \( \rho(Y,Z) \).

**ANSWER:** Using the linearity of covariance in its second argument, we find

\[ \text{Cov}(Y,Z) = \text{Cov}(Y,X) + \text{Cov}(Y,Y) \]

The first term is zero (since \( X \) and \( Y \) are independent) so this becomes \( \text{Var}(Y) = 1/12 \). The correlation coefficient is

\[
\frac{\text{Cov}(Y,Z)}{\sqrt{\text{Var}(Y)\text{Var}(Z)}} = \frac{(1/12)}{\sqrt{(1/12)(2/12)}} = 1/\sqrt{2}.
\]

6. (10 points) Let \( X \) and \( Y \) be independent exponential random variables, each with parameter \( \lambda = 5 \).

(a) Let \( f \) be the joint probability density function for the pair \((X,Y)\). Write an explicit formula for \( f \).

**ANSWER:** Since \( X \) and \( Y \) are independent, \( f(x,y) = f_X(x)f_Y(y) = 5e^{-5x} \cdot 5e^{-5y} = 25e^{-5(x+y)} \).

(b) Compute \( E[X^2Y] \).

**ANSWER:** First, note that \( X^2 \) and \( Y \) are independent, so this is \( E[X^2]E[Y] \). Direct integration gives

\[ E[Y] = 1/\lambda \text{ and } E[X^2] = 2/\lambda^2, \]

so the answer is \( 2/\lambda^3 = 2/125 \).
18.600 Probability and Random Variables
Fall 2019

For information about citing these materials or our Terms of Use, visit: https://ocw.mit.edu/terms.