18.600: Lecture 2
Multinomial coefficients and more counting problems

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Outline

Multinomial coefficients

Integer partitions

More problems
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Multinomial coefficients

Integer partitions

More problems
You have eight distinct pieces of food. You want to choose three for breakfast, two for lunch, and three for dinner. How many ways to do that?
Partition problems

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- Answer: $8!/(3!2!3!)$
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- One way to think of this: given any permutation of eight elements (e.g., 12435876 or 87625431) declare first three as breakfast, second two as lunch, last three as dinner. This maps set of 8! permutations on to the set of food-meal divisions in a many-to-one way: each food-meal division comes from 3!2!3! permutations.
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How many 8-letter sequences with 3 A’s, 2 B’s, and 3 C’s?
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How many 8-letter sequences with 3 A’s, 2 B’s, and 3 C’s?

Answer: $8!/(3!2!3!)$. Same as other problem. Imagine 8 “slots” for the letters. Choose 3 to be A’s, 2 to be B’s, and 3 to be C’s.
In general, if you have $n$ elements you wish to divide into $r$ distinct piles of sizes $n_1, n_2 \ldots n_r$, how many ways to do that?
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Answer \( \binom{n}{n_1,n_2,\ldots,n_r} := \frac{n!}{n_1!n_2!\ldots n_r!} \).
One way to understand the binomial theorem

- Expand the product \((A_1 + B_1)(A_2 + B_2)(A_3 + B_3)(A_4 + B_4)\).
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16 terms correspond to 16 length-4 sequences of \(A\)’s and \(B\)’s.

\[
\begin{align*}
A_1A_2A_3A_4 & + A_1A_2A_3B_4 + A_1A_2B_3A_4 + A_1A_2B_3B_4 + \\
A_1B_2A_3A_4 & + A_1B_2A_3B_4 + A_1B_2B_3A_4 + A_1B_2B_3B_4 + \\
B_1A_2A_3A_4 & + B_1A_2A_3B_4 + B_1A_2B_3A_4 + B_1A_2B_3B_4 + \\
B_1B_2A_3A_4 & + B_1B_2A_3B_4 + B_1B_2B_3A_4 + B_1B_2B_3B_4
\end{align*}
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\[
\begin{align*}
&A_1 A_2 A_3 A_4 + A_1 A_2 A_3 B_4 + A_1 A_2 B_3 A_4 + A_1 A_2 B_3 B_4 + \\
&A_1 B_2 A_3 A_4 + A_1 B_2 A_3 B_4 + A_1 B_2 B_3 A_4 + A_1 B_2 B_3 B_4 + \\
&B_1 A_2 A_3 A_4 + B_1 A_2 A_3 B_4 + B_1 A_2 B_3 A_4 + B_1 A_2 B_3 B_4 + \\
&B_1 B_2 A_3 A_4 + B_1 B_2 A_3 B_4 + B_1 B_2 B_3 A_4 + B_1 B_2 B_3 B_4
\end{align*}
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- What happens to this sum if we erase subscripts?
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B_1A_2A_3A_4 + B_1A_2A_3B_4 + B_1A_2B_3A_4 + B_1A_2B_3B_4 +
B_1B_2A_3A_4 + B_1B_2A_3B_4 + B_1B_2B_3A_4 + B_1B_2B_3B_4
\]

- What happens to this sum if we erase subscripts?
- \((A + B)^4 = B^4 + 4AB^3 + 6A^2B^2 + 4A^3B + A^4\). Coefficient of \(A^2B^2\) is 6 because 6 length-4 sequences have 2 \(A\)'s and 2 \(B\)'s.
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A_1A_2A_3A_4 &+ A_1A_2A_3B_4 + A_1A_2B_3A_4 + A_1A_2B_3B_4 + \\
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B_1B_2A_3A_4 &+ B_1B_2A_3B_4 + B_1B_2B_3A_4 + B_1B_2B_3B_4
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\((A + B)^4 = B^4 + 4AB^3 + 6A^2B^2 + 4A^3B + A^4\). Coefficient of \(A^2B^2\) is 6 because 6 length-4 sequences have 2 \(A\)'s and 2 \(B\)'s.

- Generally, \((A + B)^n = \sum_{k=0}^{n} \binom{n}{k} A^k B^{n-k}\), because there are \(\binom{n}{k}\) sequences with \(k\) \(A\)'s and \((n - k)\) \(B\)'s.
How about trinomials?

Expand
\[(A_1 + B_1 + C_1)(A_2 + B_2 + C_2)(A_3 + B_3 + C_3)(A_4 + B_4 + C_4).\]
How many terms?

Answer: 81, one for each length-4 sequence of A's and B's.

We can also compute \( (A + B + C)^4 = A^4 + 4A^3B + 6A^2B^2 + 4AB^3 + B^4 + 4A^3C + 12A^2BC + 12ABC^2 + 4B^3C + 6A^2C^2 + 12ABC + C^4. \)

I What is the sum of the coefficients in this expansion? What is the combinatorial interpretation of coefficient of, say, \( ABC^2 \)?

Answer: 81 = (1 + 1 + 1)^4.

\( ABC^2 \) has coefficient 12 because there are 12 length-4 words with one A, one B, two C's.
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\[(A_1 + B_1 + C_1)(A_2 + B_2 + C_2)(A_3 + B_3 + C_3)(A_4 + B_4 + C_4)\].
How many terms?

Answer: 81, one for each length-4 sequence of A’s and B’s and C’s.
What is the sum of the coefficients in this expansion? What is the combinatorial interpretation of the coefficient of, say, $ABC^2$?

Answer: $81 = (1+1+1)^4$. $ABC^2$ has coefficient 12 because there are 12 length-4 words with one $A$, one $B$, two $C$'s.

How about trinomials?

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$$(A_1 + B_1 + C_1)(A_2 + B_2 + C_2)(A_3 + B_3 + C_3)(A_4 + B_4 + C_4).$$

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Answer: 81, one for each length-4 sequence of $A$'s and $B$'s and $C$'s.

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- Answer 81 = \((1 + 1 + 1)^4\). \(ABC^2\) has coefficient 12 because there are 12 length-4 words have one \(A\), one \(B\), two \(C\)’s.
Multinomial coefficients

- Is there a higher dimensional analog of binomial theorem?
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Answer: yes.

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\[(x_1 + x_2 + \ldots + x_r)^n = \sum_{n_1, \ldots, n_r : n_1 + \ldots + n_r = n} \binom{n}{n_1, \ldots, n_r} x_1^{n_1} x_2^{n_2} \ldots x_r^{n_r}\]
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The sum on the right is taken over all collections \((n_1, n_2, \ldots, n_r)\) of \(r\) non-negative integers that add up to \(n\).
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- Pascal’s triangle gives coefficients in binomial expansions. Is there something like a “Pascal’s pyramid” for trinomial expansions?
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Pascal’s triangle gives coefficients in binomial expansions. Is there something like a “Pascal’s pyramid” for trinomial expansions?
Yes (look it up) but it is a bit trickier to draw and visualize than Pascal’s triangle.
Actually, we say $0! = 1$. What are the reasons for that?

Because there is one map from the empty set to itself.

Because we want the formula
\[
\binom{n}{k} = \frac{n!}{k!(n-k)!}
\]
to still make sense when $k = 0$ and $k = n$. There is clearly 1 way to choose $n$ elements from a group of $n$ elements. And 1 way to choose 0 elements from a group of $n$ elements so
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Because we want the recursion $n! = (n-1)!$ to hold for $n = 1$. (We won't define factorials of negative integers.)

Because we want $n!$ to equal $\int_0^\infty t^n e^{-t} dt$ to hold for all non-negative integers. (Check for positive integers by integration by parts.) This is one of those formulas you should just know. Can use it to define $n!$ for non-integer $n$.

Another common notation: write $\Gamma(z) := \int_0^\infty t^{z-1} e^{-t} dt$ and define $n! := \Gamma(n+1) = \int_0^\infty t^n e^{-t} dt$, so that $\Gamma(n) = (n-1)!$.

By the way...

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- Another common notation: Write \( \Gamma(z) := \int_0^\infty t^{z-1} e^{-t} dt \) and define \( n! := \Gamma(n+1) = \int_0^\infty t^n e^{-t} dt \), so that \( \Gamma(n) = (n-1)! \).
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How many sequences $a_1, \ldots, a_k$ of non-negative integers satisfy $a_1 + a_2 + \ldots + a_k = n$?
How many sequences $a_1, \ldots, a_k$ of non-negative integers satisfy $a_1 + a_2 + \ldots + a_k = n$?

Answer: $\binom{n+k-1}{n}$. Represent partition by $k-1$ bars and $n$ stars, e.g., as $\ast \ast \mid \ast \ast \mid \ast \ast \ast \ast \ast \ast$. 
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More counting problems

- In 18.821, a class of 27 students needs to be divided into 9 teams of three students each? How many ways are there to do that?
I

You teach a class with 90 students. In a rather severe effort to combat grade inflation, your department chair insists that you assign the students exactly 10 A's, 20 B's, 30 C's, 20 D's, and 10 F's. How many ways to do this?

\[
\binom{90}{10, 20, 30, 20, 10} = \frac{90!}{10!20!30!20!10!}
\]

You have 90 (indistinguishable) pieces of pizza to divide among the 90 (distinguishable) students. How many ways to do that (giving each student a non-negative integer number of slices)?

\[
\binom{179}{90} = \binom{179}{89}
\]

More counting problems

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\frac{27!}{(3!)^99!}
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How many 13-card bridge hands have 4 of one suit, 3 of one suit, 5 of one suit, 1 of one suit?
More counting problems

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- \[4! \left( \binom{13}{4} \right) \left( \binom{13}{3} \right) \left( \binom{13}{5} \right) \left( \binom{13}{1} \right)\]
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- How many bridge hands have at most two suits represented?
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- How many bridge hands have at most two suits represented?
  - \(\binom{4}{2} \binom{26}{13} - 8\)
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  \[ 4!(\binom{13}{4})(\binom{13}{3})(\binom{13}{5})(\binom{13}{1}) \]

- How many bridge hands have at most two suits represented?
  \[ \binom{4}{2}\binom{26}{13} - 8 \]

- How many hands have either 3 or 4 cards in each suit?

- Need three 3-card suits, one 4-card suit, to make 13 cards total. Answer is \[ 4\binom{13}{3}\binom{13}{4} \]