18.600: Lecture 24
Covariance and some conditional expectation exercises

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Covariance and correlation

Paradoxes: getting ready to think about conditional expectation
Covariance and correlation

Paradoxes: getting ready to think about conditional expectation
A property of independence

If $X$ and $Y$ are independent then

$$E[g(X)h(Y)] = E[g(X)]E[h(Y)].$$
A property of independence

- If $X$ and $Y$ are independent then
  \[ E[g(X)h(Y)] = E[g(X)]E[h(Y)]. \]
- Just write \[ E[g(X)h(Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x)h(y)f(x, y)\,dx\,dy. \]
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- Just write \[ E[g(X)h(Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x)h(y)f(x, y)\,dx\,dy. \]
- Since $f(x, y) = f_X(x)f_Y(y)$ this factors as
  \[ \int_{-\infty}^{\infty} h(y)f_Y(y)\,dy \int_{-\infty}^{\infty} g(x)f_X(x)\,dx = E[h(Y)]E[g(X)]. \]
Now define covariance of $X$ and $Y$ by
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Defining covariance and correlation

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- Note: by definition $\text{Var}(X) = \text{Cov}(X, X)$.

- Covariance (like variance) can also be written a different way. Write $\mu_X = E[X]$ and $\mu_Y = E[Y]$. If laws of $X$ and $Y$ are known, then $\mu_X$ and $\mu_Y$ are just constants.
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Then
\[
\text{Cov}(X, Y) = E[(X-\mu_X)(Y-\mu_Y)] = E[XY-\mu_X Y-\mu_Y X+\mu_X \mu_Y] = \\
E[XY] - \mu_X E[Y] - \mu_Y E[X] + \mu_X \mu_Y = E[XY] - E[X]E[Y].
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\text{Cov}(X, Y) = E[(X - \mu_X)(Y - \mu_Y)] = E[XY - \mu_X Y - \mu_Y X + \mu_X \mu_Y] = E[XY] - \mu_X E[Y] - \mu_Y E[X] + \mu_X \mu_Y = E[XY] - E[X]E[Y].
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Defining covariance and correlation

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  \[ \text{Cov}(X, Y) = E[(X - \mu_X)(Y - \mu_Y)] = E[XY - \mu_X Y - \mu_Y X + \mu_X \mu_Y] = \]  

  \[ E[XY] - \mu_X E[Y] - \mu_Y E[X] + \mu_X \mu_Y = E[XY] - E[X]E[Y]. \]

- Covariance formula \( E[XY] - E[X]E[Y] \), or “expectation of product minus product of expectations” is frequently useful.

- Note: if $X$ and $Y$ are independent then $\text{Cov}(X, Y) = 0$. 

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General statement of bilinearity of covariance:
\[
\text{Cov}(\sum_{i=1}^{m} a_i X_i, \sum_{j=1}^{n} b_j Y_j) = \sum_{i=1}^{m} \sum_{j=1}^{n} a_i b_j \text{Cov}(X_i, Y_j).
\]

Special case:
\[
\text{Var}(\sum_{i=1}^{n} X_i) = \sum_{i=1}^{n} \text{Var}(X_i) + 2 \sum_{i<j} \text{Cov}(X_i, X_j).
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- **Special case:**

  $$\text{Var} \left( \sum_{i=1}^{n} X_i \right) = \sum_{i=1}^{n} \text{Var}(X_i) + 2 \sum_{(i,j): i<j} \text{Cov}(X_i, X_j).$$
I Correlation of $X$ and $Y$ defined by
$$
\rho(X, Y) := \frac{\text{Cov}(X, Y)}{\sqrt{\text{Var}(X) \cdot \text{Var}(Y)}}.
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I Correlation doesn't care what units you use for $X$ and $Y$. If $a > 0$ and $c > 0$ then
$$
\rho(aX + b, cY + d) = \rho(X, Y).
$$

I Satisfies $-1 \leq \rho(X, Y) \leq 1$.

I Why is that? Something to do with $E[(X + Y)^2] \geq 0$ and $E[(X - Y)^2] \geq 0$?

I If $a$ and $b$ are constants and $a > 0$ then
$$
\rho(aX + b, X) = 1.
$$

I If $a$ and $b$ are constants and $a < 0$ then
$$
\rho(aX + b, X) = -1.
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- Why is that? Something to do with $E[(X + Y)^2] \geq 0$ and $E[(X - Y)^2] \geq 0$?
- If $a$ and $b$ are constants and $a > 0$ then $\rho(aX + b, X) = 1$.
- If $a$ and $b$ are constants and $a < 0$ then $\rho(aX + b, X) = -1$. 
Are independent random variables $X$ and $Y$ always uncorrelated?

Yes, assuming variances are finite (so that correlation is defined).

Are uncorrelated random variables always independent?

No. Uncorrelated just means

$$E[(X - E[X])(Y - E[Y])] = 0,$$

e.g., the outcomes where $(X - E[X])(Y - E[Y])$ is positive (the upper right and lower left quadrants, if axes are drawn centered at $(E[X], E[Y])$) balance out the outcomes where this quantity is negative (upper left and lower right quadrants). This is a much weaker statement than independence.

Important point

- Say $X$ and $Y$ are uncorrelated when $\rho(X, Y) = 0$. 

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- Suppose that $X_1, \ldots, X_n$ are i.i.d. random variables with variance 1. For example, maybe each $X_j$ takes values $\pm 1$ according to a fair coin toss.
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- Compute $\text{Cov}(X_1 + X_2 + X_3, X_2 + X_3 + X_4)$. 

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Can we generalize this example?

What is variance of number of people who get their own hat in the hat problem?

Define $X_i$ to be 1 if $i$th person gets own hat, zero otherwise.

Recall formula

$$\text{Var}(\sum_{i=1}^{n} X_i) = \sum_{i=1}^{n} \text{Var}(X_i) + 2 \sum_{i<j} \text{Cov}(X_i, X_j).$$

Reduces problem to computing $\text{Cov}(X_i, X_j)$ (for $i \neq j$) and $\text{Var}(X_i)$.

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Compute $\text{Cov}(X_1 + X_2 + X_3, X_2 + X_3 + X_4)$.

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Recall formula
\[ \Var P = \sum_{i=1}^{n} \Var X_i + 2 \sum_{i<j} \Cov(X_i, X_j). \]

Reduces problem to computing \( \Cov(X_i, X_j) \) (for \( i \neq j \)) and \( \Var X_i \).

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- Reduces problem to computing $\text{Cov}(X_i, X_j)$ (for $i \neq j$) and $\text{Var}(X_i)$. 
Outline

Covariance and correlation

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At the end of this period, a (biased) coin will be tossed. Banker will be assigned to hell forever with probability \( \frac{1}{n} \) and heaven forever with probability \( 1 - \frac{1}{n} \).

After 10 days, banker reasons, “If I wait another day I reduce my odds of being here forever from \( \frac{1}{10} \) to \( \frac{1}{11} \). That’s a reduction of \( \frac{1}{110} \). A \( \frac{1}{110} \) chance at infinity has infinite value. Worth waiting one more day.”

Repeats this reasoning every day, stays in hell forever.

Standard punch line: this is actually what banker deserved.

Fairly dark as math humor goes (and no offense intended to anyone...) but dilemma is interesting.

Famous paradox

- Certain corrupt and amoral banker dies, instructed to spend some number \( n \) (of banker’s choosing) days in hell.
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Variant without probability: Stay in hell for $n$ (of your choice) days, and thereafter on days that are multiples of $2^n$. 

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Lets me get arbitrarily rich. But if I go on forever, I return every sack given to me. If $n$th sack confers right to spend $n$th day in heaven, leads to hell-forever paradox.
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In both stories, make infinitely many good trades and end up with less than I started with.\(^{54}\) “Paradox” is existence of 2-to-1 map from (smaller set) \( \{2, 3, \ldots\} \) to (bigger set) \( \{1, 2, \ldots\} \).
You have an infinite collection of money piles with labels 0, 1, 2, ... from left to right.
Money pile paradox

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Banker seemed to make you richer (every pile got bigger) but really just reshuffled your infinite wealth.
Two envelope paradox

- $X$ is geometric with parameter $1/2$. One envelope has $10^X$ dollars, one has $10^{X-1}$ dollars. Envelopes shuffled.
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- However, “Higher expectation given amount in first envelope” may not be right notion of “better.” If $S$ is payout with switching, $T$ is payout without switching, then $S$ has same law as $T - 1$. In that sense $S$ is worse.
Two envelope paradox

VALUE OF ENVELOPE ONE

$1 (with prob. 1/4) \sim \$0.25$

$10 (with prob. 3/8) \sim \$3.75$

$100 (with prob. 3/16) \sim \$18.75$

$1000 (with prob. 3/32) \sim \$93.75$

$10000 (with prob. 3/64) \sim \$468.75$

VALUE OF ENVELOPE TWO

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Moral

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They can lead to strange conclusions, sometimes related to "reshuffling infinite (actual or expected) wealth to create more" paradoxes.

Paradoxes can arise even when total transaction is finite with probability one (as in envelope problem).

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