18.600: Lecture 34
Martingales and the optional stopping theorem

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Outline

Martingales and stopping times

Optional stopping theorem
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Optional stopping theorem
Let $X_0, X_1, X_2, \ldots$ be a sequence of random variables. Informally, we will imagine that we are acquiring information about $S$ in a sequence of stages, and each $X_j$ represents a quantity that is known to us at the $j$th stage.

If $Z$ is any random variable, we let $E[Z|F_n]$ denote the conditional expectation of $Z$ given all the information that is available to us on the $n$th stage. If we don’t specify otherwise, we assume that this information consists precisely of the values $X_0, X_1, \ldots, X_n$, so that $E[Z|F_n] = E[Z|X_0, X_1, \ldots, X_n]$.

(In some applications, one could imagine there are other things known as well at stage $n$.)

We say the sequence is a martingale if $E|X_n| < \infty$ for all $n$ and $E[X_{n+1}|F_n] = X_n$ for all $n$.

Taking into account all the information I have at stage $n$, the expected value at stage $n+1$ is the value at stage $n$. 

Martingale definition

Let $S$ be a probability space.
If $Z$ is any random variable, we let $E[Z|F_n]$ denote the conditional expectation of $X$ given all the information that is available to us on the $n$th stage. If we don’t specify otherwise, we assume that this information consists precisely of the values $X_0, X_1, \ldots, X_n$, so that $E[Z|F_n] = E[Z|X_0, X_1, \ldots, X_n]$.

(In some applications, one could imagine there are other things known as well at stage $n$.)

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- If $Z$ is any random variable, we let $E[Z|\mathcal{F}_n]$ denote the conditional expectation of $X$ given all the information that is available to us on the $n$th stage. If we don't specify otherwise, we assume that this information consists precisely of the values $X_0, X_1, \ldots, X_n$, so that $E[Z|\mathcal{F}_n] = E[Z|X_0, X_1, \ldots, X_n]$. (In some applications, one could imagine there are other things known as well at stage $n$.)
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(In some applications, one could imagine there are other things known as well at stage $n$.)

We say $X_n$ sequence is a **martingale** if $E[|X_n|] < \infty$ for all $n$ and $E[X_{n+1}|\mathcal{F}_n] = X_n$ for all $n$. 
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- If $Z$ is any random variable, we let $E[Z|\mathcal{F}_n]$ denote the conditional expectation of $X$ given all the information that is available to us on the $n$th stage. If we don’t specify otherwise, we assume that this information consists precisely of the values $X_0, X_1, \ldots, X_n$, so that $E[Z|\mathcal{F}_n] = E[Z|X_0, X_1, \ldots, X_n]$. (In some applications, one could imagine there are other things known as well at stage $n$.)
- We say $X_n$ sequence is a martingale if $E[|X_n|] < \infty$ for all $n$ and $E[X_{n+1}|\mathcal{F}_n] = X_n$ for all $n$.
- “Taking into account all the information I have at stage $n$, the expected value at stage $n+1$ is the value at stage $n$.”
Example: Imagine that $X_n$ is the price of a stock on day $n$. 
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Question: If you are given a mathematical description of a process $X_0, X_1, X_2, \ldots$ then how can you check whether it is a martingale?
Example: Imagine that $X_n$ is the price of a stock on day $n$.

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Question: If you are given a mathematical description of a process $X_0, X_1, X_2, \ldots$ then how can you check whether it is a martingale?

Consider all of the information that you know after having seen $X_0, X_1, \ldots, X_n$. Then try to figure out what additional (not yet known) randomness is involved in determining $X_{n+1}$. Use this to figure out the conditional expectation of $X_{n+1}$, and check to see whether this is necessarily equal to the known $X_n$ value.
Let \( X_0 = 0 \) and \( X_n = P \sum_{i=1}^n A_i \) for \( n > 0 \). Is the \( X_n \) sequence a martingale?

Answer: yes. To see this, note that

\[
E[X_{n+1} | F_n] = E[X_n + A_{n+1} | F_n] = E[X_n | F_n] + E[A_{n+1} | F_n],
\]

by additivity of conditional expectation (given \( F_n \)).

Since \( X_n \) is known at stage \( n \), we have

\[
E[X_n | F_n] = X_n.
\]

Since we know nothing more about \( A_{n+1} \) at stage \( n \) than we originally knew, we have

\[
E[A_{n+1} | F_n] = 0.
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Thus

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Informally, I'm just tossing a new fair coin at each stage to see if \( X_n \) goes up or down one step. If I know the information available up to stage \( n \), and I know \( X_n = 10 \), then I see \( X_{n+1} = 11 \) and \( X_{n+1} = 9 \) as equally likely, so

\[
E[X_{n+1} | F_n] = 10 = X_n.
\]

### Martingale examples

- Suppose that \( A_1, A_2, \ldots \) are i.i.d. random variables each equal to \(-1\) with probability .5 and 1 with probability .5.
Suppose that $A_1, A_2, \ldots$ are i.i.d. random variables each equal to $-1$ with probability $0.5$ and $1$ with probability $0.5$.

Let $X_0 = 0$ and $X_n = \sum_{i=1}^{n} A_i$ for $n > 0$. Is the $X_n$ sequence a martingale?
Since $X_n$ is known at stage $n$, we have $E[X_n|F_n] = X_n$. Since we know nothing more about $A_{n+1}$ at stage $n$ than we originally knew, we have $E[A_{n+1}|F_n] = 0$. Thus $E[X_{n+1}|F_n] = X_n$.

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Martingale examples

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Answer: yes. To see this, note that

$$E[X_{n+1}|\mathcal{F}_n] = E[X_n + A_{n+1}|\mathcal{F}_n] = E[X_n|\mathcal{F}_n] + E[A_{n+1}|\mathcal{F}_n],$$

by additivity of conditional expectation (given $\mathcal{F}_n$).
Martingale examples

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- Answer: yes. To see this, note that
  \[ E[X_{n+1}|\mathcal{F}_n] = E[X_n + A_{n+1}|\mathcal{F}_n] = E[X_n|\mathcal{F}_n] + E[A_{n+1}|\mathcal{F}_n], \]
  by additivity of conditional expectation (given $\mathcal{F}_n$).
- Since $X_n$ is known at stage $n$, we have $E[X_n|\mathcal{F}_n] = X_n$. Since we know nothing more about $A_{n+1}$ at stage $n$ than we originally knew, we have $E[A_{n+1}|\mathcal{F}_n] = 0$. Thus $E[X_{n+1}|\mathcal{F}_n] = X_n$. 

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Martingale examples

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  \[ E[X_{n+1}|\mathcal{F}_n] = E[X_n + A_{n+1}|\mathcal{F}_n] = E[X_n|\mathcal{F}_n] + E[A_{n+1}|\mathcal{F}_n], \]
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- Since $X_n$ is known at stage $n$, we have $E[X_n|\mathcal{F}_n] = X_n$. Since we know nothing more about $A_{n+1}$ at stage $n$ than we originally knew, we have $E[A_{n+1}|\mathcal{F}_n] = 0$. Thus
  \[ E[X_{n+1}|\mathcal{F}_n] = X_n. \]

- Informally, I’m just tossing a new fair coin at each stage to see if $X_n$ goes up or down one step. If I know the information available up to stage $n$, and I know $X_n = 10$, then I see $X_{n+1} = 11$ and $X_{n+1} = 9$ as\(^7\) equally likely, so
  \[ E[X_{n+1}|\mathcal{F}_n] = 10 = X_n. \]
Another martingale example

- What if each $A_i$ is 1.01 with probability .5 and .99 with probability .5 and we write $X_0 = 1$ and $X_n = \prod_{i=1}^n A_i$ for $n > 0$? Then is $X_n$ a martingale?
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Answer: yes. Note that $E[X_{n+1}|\mathcal{F}_n] = E[A_{n+1}X_n|\mathcal{F}_n]$. At stage $n$, the value $X_n$ is known, and hence can be treated as a known constant, which can be factored out of the expectation, i.e., $E[A_{n+1}X_n|\mathcal{F}_n] = X_n E[A_{n+1}|\mathcal{F}_n]$. 

Another martingale example
Since I know nothing new about $A_{n+1}$ at stage $n$, we have 

$$E[A_{n+1}|F_n] = E[A_{n+1}] = 1.$$ 

Hence 

$$E[A_{n+1}X_n|F_n] = X_n.$$

Informally, I’m just tossing a new fair coin at each stage to see if $X_n$ goes up or down by a percentage point of its current value. If I know all the information available up to stage $n$, and I know $X_n = 5$, then I see $X_{n+1} = 5.05$ and $X_{n+1} = 4.95$ as equally likely, so $E[X_{n+1}|F_n] = 5$.

Two classic martingale examples:

- Sums of independent random variables (each with mean zero).
- Products of independent random variables (each with mean one).

Another martingale example

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- **Two classic martingale examples**: sums of independent random variables (each with mean zero) and products of independent random variables (each with mean one).
I What is \( E[X_n] \), as a function of \( n \)?

I \( E[X_n] = 0 \) for all \( n \).

I Does this mean that \( X_n \) is a martingale?

I No. If \( n \geq 1 \), then given the information available up to stage \( n \), I can figure out what \( A \) must be, and can hence deduce exactly what \( X_{n+1} \) will be — and it is not the same as \( X_n \). In particular, \( E[X_{n+1} | F_n] = -X_n \).

I Informally, \( X_n \) alternates between 1 and \(-1\). Each time it goes up and hits 1, I know it will go back down to \(-1\) on the next step.

Another example

- Suppose \( A \) is 1 with probability .5 and \(-1\) with probability .5. Let \( X_0 = 0 \) and write \( X_n = (-1)^n A \) for all \( n > 0 \).
Another example

- Suppose $A$ is 1 with probability $0.5$ and $-1$ with probability $0.5$. Let $X_0 = 0$ and write $X_n = (-1)^n A$ for all $n > 0$.
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- No. If $n \geq 1$, then given the information available up to stage $n$, I can figure out what $A$ must be, and can hence deduce exactly what $X_{n+1}$ will be — and it is not the same as $X_n$. In particular, $E[X_{n+1} | \mathcal{F}_n] = -X_n \neq X_n$. 
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- Informally, \( X_n \) alternates between 1 and \(-1\). Each time it goes up and hits 1, I know it will go back down to \(-1\) on the next step.
Let $T$ be a non-negative integer valued random variable.
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Think of $T$ as giving the time the asset will be sold if the price sequence is $X_0, X_1, X_2, \ldots$.

Say that $T$ is a stopping time if the event that $T = n$ depends only on the values $X_i$ for $i \leq n$. In other words, the decision to sell at time $n$ depends only on prices up to time $n$, not on (as yet unknown) future prices.
Which of the following is a stopping time?

1. The smallest \( T \) for which \( |X_T| = 50 \)
2. The smallest \( T \) for which \( X_T \in \{-10, 100\} \)
3. The smallest \( T \) for which \( X_T = 0 \).
4. The \( T \) at which the \( X_n \) sequence achieves the value 17 for the 9th time.
5. The value of \( T \in \{0, 1, 2, \ldots, 100\} \) for which \( X_T \) is largest.
6. The largest \( T \in \{0, 1, 2, \ldots, 100\} \) for which \( X_T = 0 \).

Answer: first four, not last two.

Let \( A_1, \ldots \) be i.i.d. random variables equal to \(-1\) with probability .5 and 1 with probability .5 and let \( X_0 = 0 \) and \( X_n = \sum_{i=1}^{n} A_i \) for \( n \geq 0 \).
Let $A_1, \ldots$ be i.i.d. random variables equal to $-1$ with probability $0.5$ and $1$ with probability $0.5$ and let $X_0 = 0$ and $X_n = \sum_{i=1}^{n} A_i$ for $n \geq 0$.

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Answer: first four, not last two.
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Optional stopping theorem
Essentially says that you can't make money (in expectation) by buying and selling an asset whose price is a martingale. Precisely, if you buy the asset at some time and adopt any strategy at all for deciding when to sell it, then the expected price at the time you sell is the price you originally paid. If market price is a martingale, you cannot make money in expectation by "timing the market."

Doob’s optional stopping time theorem is contained in many basic texts on probability and Martingales. (See, for example, Theorem 10.10 of *Probability with Martingales*, by David Williams, 1991.)
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Precisely, if you buy the asset at some time and adopt any strategy at all for deciding when to sell it, then the expected price at the time you sell is the price you originally paid.

If market price is a martingale, you cannot make money in expectation by “timing the market.”
When we say martingale is bounded, we mean that for some $C$, we have that with probability one $|X_i| < C$ for all $i$.

Why is this assumption necessary? Can we give a counterexample if boundedness is not assumed?

Theorem can be proved by induction if stopping time $T$ is bounded. Unbounded $T$ requires a limit argument. (This is where boundedness of martingale is used.)

Doob’s Optional Stopping Theorem: If the sequence $X_0, X_1, X_2, \ldots$ is a **bounded** martingale, and $T$ is a stopping time, then the expected value of $X_T$ is $X_0$. 

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- **Doob’s Optional Stopping Theorem**: If the sequence $X_0, X_1, X_2, \ldots$ is a **bounded** martingale, and $T$ is a stopping time, then the expected value of $X_T$ is $X_0$.

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Doob’s Optional Stopping Theorem: statement

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- Theorem can be proved by induction if *stopping time* $T$ is bounded. Unbounded $T$ requires a limit argument. (This is where boundedness of martingale is used.)
Efficient market hypothesis: new information is instantly absorbed into the stock value, so expected value of the stock tomorrow should be the value today. (If it were higher, statistical arbitrageurs would bid up today's price until this was not the case.)

But what about interest, risk premium, etc.?

According to the fundamental theorem of asset pricing, the discounted price $X^n A^n$, where $A^n$ is a risk-free asset, is a martingale with respected to risk neutral probability. More on this next lecture.

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- Depends only on $Y$. Describes expectation of $X$ given observed $Y$ value.
- We showed $E[E[X|Y]] = E[X]$.
- This means that the three-element sequence $E[X], E[X|Y], X$ is a martingale.

Martingales as successively revised best guesses

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More generally if $Y_i$ are any random variables, the sequence $E[X], E[X|Y_1], E[X|Y_1, Y_2], E[X|Y_1, Y_2, Y_3], \ldots$ is a martingale.
Ivan sees email from girlfriend with subject “some possibly serious news”, thinks there’s a 20 percent chance she’ll break up with him by email’s end. Revises number after each line:
I have something crazy to tell you, and so sorry to do this by email. (Where’s your phone!?) I’ve been spending lots of time with a guy named Robert, a visiting database consultant on my project who seems very impressed by my work. Robert wants me to join his startup in Palo Alto. Exciting!!! Of course I said I’d have to talk to you first, because you are absolutely my top priority in my life, and you’re stuck at MIT for at least three more years... but honestly, I’m just so confused on so many levels. Call me!!! I love you! Alice

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Martingales as real-time subjective probability updates
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Example: let $C$ be the amount of oil available for drilling under a particular piece of land. Suppose that ten geological tests are done that will ultimately determine the value of $C$. Let $C_n$ be the **conditional expectation** of $C$ given the outcome of the first $n$ of these tests. Then the sequence $C_0, C_1, C_2, \ldots, C_{10} = C$ is a martingale.
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Let $A_i$ be my best guess at the probability that a basketball team will win the game, given the outcome of the first $i$ minutes of the game. Then (assuming some “rationality” of my personal probabilities) $A_i$ is a martingale.