18.600: Lecture 35
Martingales and risk neutral probability

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MIT
Martingales and stopping times

Martingales and Bayesian expectation revisions

Risk neutral probability and martingales
Outline

Martingales and stopping times

Martingales and Bayesian expectation revisions

Risk neutral probability and martingales
Recall martingale definition

Let $S$ be the probability space. Let $X_0, X_1, X_2, \ldots$ be a sequence of real random variables. Interpret $X_i$ as price of asset at $i$th time step.
Recall martingale definition

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- Say $X_n$ sequence is a **martingale** if $E[|X_n|] < \infty$ for all $n$ and $E[X_{n+1}|\mathcal{F}_n] = X_n$ for all $n$. 
Recall martingale definition

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- “Given all I know today, expected price tomorrow is the price today.”
Recall stopping time definition

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Think of $T$ as giving the time the asset will be sold if the price sequence is $X_0, X_1, X_2, \ldots$.

Say that $T$ is a **stopping time** if the event that $T = n$ depends only on the values $X_i$ for $i \leq n$. In other words, the decision to sell at time $n$ depends only on prices up to time $n$, not on (as yet unknown) future prices.
Examples

- Suppose that an asset price is a martingale that starts at 50 and changes by increments of ±1 at each time step. What is the probability that the price goes down to 40 before it goes up to 70?
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- What is the probability that it goes down to 45 then up to 55 then down to 45 then up to 55 again — all before reaching either 0 or 100?
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Risk neutral probability and martingales
In previous lectures, we interpreted the conditional expectation $E[X|Y]$ as a random variable. It depends only on $Y$. Describes expectation of $X$ given the observed $Y$ value.

We showed $E[E[X|Y]] = E[X]$. This means that the three-element sequence $E[X], E[X|Y], X$ is a martingale.

More generally, $E[X|F_0], E[X|F_1], E[X|F_2], \ldots$ is a martingale.

Martingales as successively revised best guesses

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This means that the three-element sequence \( E[X], E[X|Y], X \) is a martingale.

More generally, \( E[X|\mathcal{F}_0], E[X|\mathcal{F}_1], E[X|\mathcal{F}_2], \ldots \) is a martingale,
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This is \textit{not} a statement about how well informed my probability measure is.
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I have something crazy to tell you, and so sorry to do this by email. (Where's your phone!?)

I've been spending lots of time with a guy named Robert, a visiting database consultant on my project who seems very impressed by my work. Robert wants me to join his startup in Palo Alto. Exciting!!! Of course I said I'd have to talk to you first, because you are absolutely a priority in my life, and you'll be at MIT for at least three more years... but I'm just so confused on so many levels. Call me!!! I love you! Alice

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Martingales as real time subjective probability estimates

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Risk neutral probability and martingales
Many asset prices are believed to behave approximately like martingales, at least in the short term.
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According to the **fundamental theorem of asset pricing**, the discounted price \( \frac{X(n)}{A(n)} \), where \( A \) is a risk-free asset, is a martingale with respected to **risk neutral probability**.
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For example, suppose somebody is about to shoot a free throw in basketball. What is the price in the sports betting world of a contract that pays one dollar if the shot is made?

If the answer is .75 dollars, then we say that the risk neutral probability that the shot will be made is .75.
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Risk neutral probability is the probability determined by the market betting odds.
Risk neutral probability of outcomes known at fixed time $T$

- **Risk neutral probability of event $A$:** $P_{RN}(A)$ denotes

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\frac{\text{Price}\{\text{Contract paying 1 dollar at time } T \text{ if } A \text{ occurs}\}}{\text{Price}\{\text{Contract paying 1 dollar at time } T \text{ no matter what}\}}.
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- Assuming no arbitrage (i.e., no risk-free profit with zero upfront investment), $P_{RN}$ satisfies axioms of probability. That is, $0 \leq P_{RN}(A) \leq 1$, and $P_{RN}(S) = 1$, and if events $A_j$ are disjoint then $P_{RN}(A_1 \cup A_2 \cup \ldots) = P_{RN}(A_1) + P_{RN}(A_2) + \ldots$.
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- **Arbitrage example**: if $A$ and $B$ are disjoint and $P_{RN}(A \cup B) < P(A) + P(B)$ then we sell contracts paying 1 if $A$ occurs and 1 if $B$ occurs, buy contract paying 1 if $A \cup B$ occurs, pocket difference. 52
But suppose $A$ is the event that the government is dissolved and all dollars become worthless. What is $P_{RN}(A)$?

$P_{RN}(A)$ should be 0. Even if people think $A$ is likely, a contract paying a dollar when $A$ occurs is worthless.

Now, suppose there are only 2 outcomes: $A$ is the event that economy booms and everyone prospers, and $B$ is the event that economy sags and everyone is needy. Suppose purchasing power of dollar is the same in both scenarios. If people think $A$ has a 0.5 chance to occur, do we expect $P_{RN}(A) > 0.5$ or $P_{RN}(A) < 0.5$?

Answer: $P_{RN}(A) < 0.5$. People are risk averse. In the second scenario they need the money more.

Risk neutral probability differ vs. “ordinary probability”

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Answer: $P_{RN}(A) < .5$. People are risk averse. In second scenario they need the money more.
Non-systemic event

Suppose that $A$ is the event that the Boston Red Sox win the World Series. Would we expect $P_{RN}(A)$ to represent (the market’s best assessment of) the probability that the Red Sox will win?
Even if some people bet based on loyalty, emotion, insurance against personal financial exposure to team’s prospects, etc., there will arguably be enough in-it-for-the-money statistical arbitrageurs to keep price near a reasonable guess of what well-informed informed experts would consider the true probability.

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Before the 2016 US presidential election, investors predicted (correctly) that the value of the Mexican peso (in US dollars) would be substantially lower if Trump won than if Clinton won.

Given this, would the risk neutral probability of a Trump win have been higher with pesos as the numéraire or with dollars as the numéraire?

Risk neutral probability can be defined for variable times and variable interest rates — e.g., one can take the numéraire to be amount one dollar in a variable-interest-rate money market account has grown to when outcome is known. Can define $P_{RN}(A)$ to be price of contract paying this amount if and when $A$ occurs.

For simplicity, we focus on fixed time $T$, fixed interest rate $r$ in this lecture.

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- Risk neutral probability can be defined for variable times and variable interest rates — e.g., one can take the numéraire to be amount one dollar in a variable-interest-rate money market account has grown to when outcome is known. Can define $P_{RN}(A)$ to be price of contract paying this amount if and when $A$ occurs.
- For simplicity, we focus on fixed time $T$, fixed interest rate $r$ in this lecture.
Many financial derivatives are essentially bets of this form. Unlike "true probability" (what does that mean?) the "risk neutral probability" is an objectively measurable price.

Pundit: The market predictions are ridiculous. I can estimate probabilities much better than they can.

Listener: Then why not make some bets and get rich? If your estimates are so much better, law of large numbers says you'll surely come out way ahead eventually.

Pundit: Well, you know... been busy... scruples about gambling... more to life than money...

Listener: Yeah, that's what I thought.

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If $r$ is risk free interest rate, then by definition, price of a contract paying dollar at time $T$ if $A$ occurs is $P_{RN}(A)e^{-rT}$.
Prices as expectations

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If $A$ and $B$ are disjoint, what is the price of a contract that pays 2 dollars if $A$ occurs, 3 if $B$ occurs, 0 otherwise?

Answer: $(2P_{RN}(A) + 3P_{RN}(B))e^{-rT}$. 
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- Implies **fundamental theorem of asset pricing**, which says discounted price $X(n)A(n)$ (where $A$ is a risk-free asset) is a martingale with respected to risk neutral probability.