18.600: Lecture 36
Call functions and Black-Scholes

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MIT
Outline

Call function

Black-Scholes
Call function

Black-Scholes
Call function: pretty cool whether you love finance or not

- **Recall:** if $X$ is non-negative random variable with cumulative distribution function $F$, then $\int_0^\infty (1 - F(x)) \, dx = E[X]$. 

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What is the meaning of $C(K) := \int_{K}^{\infty} (1 - F(x)) \, dx$?

It is the area bounded between $y = F(x)$ and $y = 1$ and $x = K$.

By translation argument, it is also $E[\max(X - K, 0)]$.

Note: $C_0(x) = -1 + F(x) = F(x) - 1$ and $C_00(x) = f(x)$.

Let's give $C$ a name: we'll call it the call function of $X$.

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2. $C(K)$ is area between $y = F(x)$ and $y = 1$ and $x = K$.
3. $C(K)$ is an anti-anti-derivative of the density function $f$.

Note that $C(0) = E[X]$ and $\lim_{K \to \infty} C(K) = 0$.

$C$ is convex with slope increasing from $-1$ to $0$.

So now any random variable $X$ comes with a pdf $f_X$, a cdf $F_X$ (an anti-derivative of $f_X$) and this call function $C = C_X$ (an anti-anti-derivative of $f_X$).

Wonder if $C$ is good for anything....

Recall: if $X$ is non-negative random variable with cumulative distribution function $F$, then $\int_{0}^{\infty} (1 - F(x)) \, dx = E[X]$.

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- Wonder if $C$ is good for anything....
Math goal: understand \( C \) and how to compute it the special case that \( X = e^N \), where \( N \) is a normal random variable.

Story goal: give some financial motivation for all of this. Explain what \( C \) has to do with option pricing and what the special case \( X = e^N \) has to do with the Black-Scholes formula.

Weird fact: If \( X \) is a real world random quantity (such as the price of gold or euros or stock shares at a future date) and we use risk neutral probability, then sometimes the call function \( C \) (or a related “put function”) is what we can look up online. One then uses the quoted \( C \) values to work out \( F_X \) and \( f_X \).

Grand story goal: Say something about the link between probability and the real world. What is the probability that the price of Microsoft stock will rise by more than ten dollars over the next month? What is the probability that the price of oil will drop more than ten percent next year? How can I (using internet and math) come up with a reasonable answer?

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If $r$ is risk free interest rate, then by definition, price of a contract paying dollar at time $T$ if $A$ occurs is $P_{RN}(A)e^{-rT}$. 

Asset price as discounted expectation: $X_0$ $E_{RN}(X_T)e^{-rT}$
If $r$ is risk free interest rate, then by definition, price of a contract paying dollar at time $T$ if $A$ occurs is $P_{RN}(A)e^{-rT}$.

If $A$ and $B$ are disjoint, what is the price of a contract that pays 2 dollars if $A$ occurs, 3 if $B$ occurs, 0 otherwise?
I Generally, in absence of arbitrage, price of contract that pays \( X \) at time \( T \) should be 
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E_{RN}(X_T) e^{-rT}
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where \( E_{RN} \) denotes expectation with respect to the risk neutral probability.

I Example: if a non-dividend-paying stock will be worth \( X \) at time \( T \), then its price today should be 
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I Risk neutral probability basically defined so price of asset today is 
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e^{-rT} \text{times risk neutral expectation of time } T \text{ price.}
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I In particular, the risk neutral expectation of tomorrow’s (interest discounted) stock price is today’s stock price.

I Implies fundamental theorem of asset pricing, which says discounted price \( X^n(A) \) where \( A \) is a risk-free asset is a martingale with respected to risk neutral probability.

If \( r \) is risk free interest rate, then by definition, price of a contract paying dollar at time \( T \) if \( A \) occurs is \( P_{RN}(A) e^{-rT} \).

If \( A \) and \( B \) are disjoint, what is the price of a contract that pays 2 dollars if \( A \) occurs, 3 if \( B \) occurs, 0 otherwise?

Answer: \((2P_{RN}(A) + 3P_{RN}(B)) e^{-rT}\).
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European call options

A **European call option** on a stock at **maturity date** $T$, **strike price** $K$, gives the holder the right (but not obligation) to purchase a share of stock for $K$ dollars at time $T$.

The document gives the bearer the right to purchase one share of MSFT from me on May 31 for 35 dollars. SS
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  where $C$ is the call function corresponding to $X$. 

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- Can look up $C(K)$ values for stock (say GOOG) at.cboe.com, apply smoothing, take derivatives, approximate $F_X$ and $f_X$. 
European put options

- **European put option** gives holder write to sell stock for $K$ dollars at time $T$. 

  $$P(K) = E \left[ \max(K - X, 0) \right]$$

  $$C(K) - P(K) = E[X - K].$$

  $$P(K) = C(K) - E[X] + K \int_0^T F(x) \, dx.$$
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- Analysis is basically the same as for call options except that one replaces the “call function” $C(K) = E[\max(X - K, 0)]$ with the “put function” defined by

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- For simplicity we focus on call functions in this lecture.
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Black-Scholes: main assumption and conclusion

- More famous MIT professors: Black, Scholes, Merton.
Black-Scholes: main assumption and conclusion

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- 1997 Nobel Prize.
Observation: Normal ($\mu, T\sigma^2$) implies $E[e^N] = e^{\mu + \frac{T\sigma^2}{2}}$.

Observation: If $X_0$ is the current price then $X_0 = E[R^N[X]]e^{-rT} = E[e^N]e^{-rT} = e^{\mu + \frac{(\sigma^2 - r)T}{2}}$.

Observation: This implies $\mu = \log X_0 + (r - \frac{\sigma^2}{2})T$.

General Black-Scholes conclusion: If $g$ is any function then the price of a contract that pays $g(X)$ at time $T$ is $E[g(e^N)]e^{-rT}$ where $N$ is normal with mean $\mu$ and variance $T\sigma^2$.

Surprise: No need to guess $\mu$. It is fixed by $X_0$, $r$, $\sigma$, $T$.

Black-Scholes: main assumption and conclusion

- More famous MIT professors: Black, Scholes, Merton.
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- Assumption: the log of an asset price $X$ at fixed future time $T$ is a normal random variable (call it $N$) with some known variance (call it $T\sigma^2$) and some mean (call it $\mu$) with respect to risk neutral probability.
Observation: If \( X_0 \) is the current price then
\[
X_0 = E[RN][X_{e^{-rT}}] = e^{\mu + (\sigma^2/2 - r)T}.
\]
Observation: This implies \( \mu = \log X_0 + (r - \sigma^2/2)T \).

General Black-Scholes conclusion: If \( g \) is any function then the price of a contract that pays \( g(X) \) at time \( T \) is
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E[\exp(N_{e^{-rT}})] = e^{\mu + T\sigma^2/2}.
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Black-Scholes: main assumption and conclusion

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1997 Nobel Prize.

Assumption: the log of an asset price \( X \) at fixed future time \( T \) is a normal random variable (call it \( N \)) with some known variance (call it \( T\sigma^2 \)) and some mean (call it \( \mu \)) with respect to risk neutral probability.

Observation: \( N \) normal \((\mu, T\sigma^2)\) implies \( E[e^N] = e^{\mu + T\sigma^2/2} \).
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**General Black–Scholes conclusion:** If $g$ is any function then the price of a contract that pays $g(X)$ at time $T$ is

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**Surprise:** No need to guess $\mu$. It is fixed by $X_0$, $r$, $\sigma$, $T$. 

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Black-Scholes for European call option

A **European call option** on a stock at maturity date \( T \), strike price \( K \), gives the holder the right (but not obligation) to purchase a share of stock for \( K \) dollars at time \( T \).

The document gives the bearer the right to purchase one share of MSFT from me on May 31 for 35 dollars. **SS**
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Recall: If $X$ is time $T$ stock price, then value of option at time $T$ is $g(X) = \max\{0, X - K\}$. Price now should be

$$e^{-rT} \mathbb{E}_{R_{N}} g(X) = e^{-rT} C(K).$$
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Write this as

$$e^{-rT} E[\max\{0, e^N - K\}] = e^{-rT} E[(e^N - K) 1_{N \geq \log K}]$$

$$= \frac{e^{-rT}}{\sigma \sqrt{2\pi T}} \int_{\log K}^{\infty} e^{-\frac{(x-\mu)^2}{2T\sigma^2}} (e^x - K) \, dx.$$
We need to compute
\[ e^{-rT \sigma \sqrt{2\pi T R \infty \log K} e^{-(x - \mu)^2/2T \sigma^2}} (e^x - K) \]
dx
where 
\[ \mu = rT + \log X_0 - T \sigma^2/2. \]

I can use complete-the-square tricks to compute the two terms explicitly in terms of standard normal cumulative distribution function \( \Phi \).

Price of European call is \( \Phi(d_1) X_0 - \Phi(d_2) Ke^{-rT} \) where 
\[ d_1 = \ln(X_0/K) + (r + \sigma^2/2)(T) \sigma \sqrt{T} \]
and
\[ d_2 = \ln(X_0/K) + (r - \sigma^2/2)(T) \sigma \sqrt{T}. \]

The famous formula

- Let \( T \) be time to maturity, \( X_0 \) current price of underlying asset, \( K \) strike price, \( r \) risk free interest rate, \( \sigma \) the volatility.
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Risk neutral probability densities derived from call quotes are not quite lognormal in practice. Tails are too fat. Main Black-Scholes assumption is only approximately correct.
Perspective: implied volatility

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- “Implied volatility” is the value of $\sigma$ that (when plugged into Black-Scholes formula along with known parameters) predicts the current market price.
- If Black-Scholes were completely correct, then given a stock and an expiration date, the implied volatility would be the same for all strike prices $K$. In practice, when the implied volatility is viewed as a function of $K$ (sometimes called the “volatility smile”), it is not constant.
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Nonetheless, “implied volatility” has become a standard part of the finance lexicon. When traders want to get a rough sense of how a financial derivative is priced, they often ask for the implied volatility (a number automatically computed in many financial software packages).
Perspective: why is Black-Scholes not exactly right?

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- **Replicating portfolio point of view:** in simple models (e.g., where wealth always goes up or down by fixed factor each day) can transfer money between the stock and the risk free asset to ensure our wealth at time $T$ equals option payout. Option price is required initial investment, which is risk neutral expectation of payout.
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- **Where arguments for assumption break down:** Fluctuation sizes vary from day to day. Prices can have big jumps. Past volatility does not determine future volatility.
- **Fixes:** variable volatility, random interest rates, Lévy jumps....