Eight athletic teams are ranked 1 through 8 after season one, and ranked 1 through 8 again after season two. Assume that each set of rankings is chosen uniformly from the set of 8! possible rankings and that the two rankings are independent. Let $N$ be the number of teams whose rank does not change from season one to season two. Let $N_+$ the number of teams whose rank improves by exactly two spots. Let $N_-$ be the number whose rank declines by exactly two spots. Compute the following:
Expectation and variance

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- $E[N]$, $E[N_+]$, and $E[N_-]$
- $\text{Var}[N]$
- $\text{Var}[N_+]$
Let $N_i$ be 1 if team ranked $i$th first season remains $i$th second seasons. Then $E[N] = E\left[\sum_{i=1}^{8} N_i\right] = 8 \cdot \frac{1}{8} = 1$. Similarly, $E[N_+] = E[N_-] = 6 \cdot \frac{1}{8} = 3/4$. 
Let $N_i$ be 1 if team ranked $i$th first season remains $i$th second seasons. Then $\mathbb{E}[N] = \mathbb{E}[\sum_{i=1}^{8} N_i] = 8 \cdot \frac{1}{8} = 1$. Similarly, $\mathbb{E}[N_+] = \mathbb{E}[N_-] = 6 \cdot \frac{1}{8} = 3/4$

$\text{Var}[N] = \mathbb{E}[N^2] - \mathbb{E}[N]^2$ and $\mathbb{E}[N^2] = \mathbb{E}[\sum_{i=1}^{8} \sum_{j=1}^{8} N_i N_j] = 8 \cdot \frac{1}{8} + 56 \cdot \frac{1}{56} = 2$. 

Expectation and variance answers
- Let $N_i$ be 1 if team ranked $i$th first season remains $i$th second seasons. Then $E[N] = E[\sum_{i=1}^{8} N_i] = 8 \cdot \frac{1}{8} = 1$. Similarly, $E[N_+] = E[N_-] = 6 \cdot \frac{1}{8} = 3/4$.

- $\text{Var}[N] = E[N^2] - E[N]^2$ and $E[N^2] = E[\sum_{i=1}^{8} \sum_{j=1}^{8} N_i N_j] = 8 \cdot \frac{1}{8} + 56 \cdot \frac{1}{56} = 2$.

- $N^i_+$ be 1 if team ranked $i$th has rank improve to $(i-2)$th for second seasons. Then $E[(N_+)^2] = E[\sum_{3=1}^{8} \sum_{3=1}^{8} N^i_+ N^{j}_+] = 6 \cdot \frac{1}{8} + 30 \cdot \frac{1}{56} = 9/7$, so $\text{Var}[N_+] = 9/7 - (3/4)^2$. 


Roll ten dice. Find the conditional probability that there are exactly 4 ones, given that there are exactly 4 sixes.
Numerator: is \((10^4)(6^4)42^610\). Denominator is \((10^4)56^610\).

\[
\begin{align*}
\text{Ratio is } &\frac{6^442}{56} = \frac{6^4(154)}{462}.
\end{align*}
\]

Alternate solution: first condition on location of the 6's and then use binomial theorem.

Conditional distributions answers

- Straightforward approach: \(P(A|B) = \frac{P(AB)}{P(B)}\).
Conditional distributions  answers

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- Numerator: is \( \frac{\binom{10}{4}\binom{6}{4}4^2}{6^{10}} \). Denominator is \( \frac{\binom{10}{4}5^6}{6^{10}} \).
Conditional distributions answers

- Straightforward approach: \( P(A|B) = \frac{P(AB)}{P(B)} \).
- Numerator: is \( \binom{10}{4} \binom{6}{4} 4^2 \). Denominator is \( \binom{10}{4} 5^6 \).
- Ratio is \( \binom{6}{4} 4^2 / 5^6 = \binom{6}{4} \left(\frac{1}{5}\right)^4 \left(\frac{4}{5}\right)^2 \).
Conditional distributions

- Straightforward approach: \( P(A|B) = P(AB)/P(B) \).
- Numerator: is \( \frac{(10)(6)4^2}{6^{10}} \). Denominator is \( \frac{(10)5^6}{6^{10}} \).
- Ratio is \( \frac{(6)4^2}{5^6} = \left(\frac{6}{4}\right)^4 \left(\frac{4}{5}\right)^2 \).
- Alternate solution: first condition on location of the 6’s and then use binomial theorem.
Suppose that in a certain town earthquakes are a Poisson point process, with an average of one per decade, and volcano eruptions are an independent Poisson point process, with an average of two per decade. The $V$ be length of time (in decades) until the first volcano eruption and $E$ the length of time (in decades) until the first earthquake. Compute the following:

- $\mathbb{E}[E^2]$ and $\text{Cov}[E, V]$. 
Poisson point processes

Suppose that in a certain town earthquakes are a Poisson point process, with an average of one per decade, and volcano eruptions are an independent Poisson point process, with an average of two per decade. The $V$ be length of time (in decades) until the first volcano eruption and $E$ the length of time (in decades) until the first earthquake. Compute the following:

- $\mathbb{E}[E^2]$ and $\text{Cov}[E, V]$.
- The expected number of calendar years, in the next decade (ten calendar years), that have no earthquakes and no volcano eruptions.
Suppose that in a certain town earthquakes are a Poisson point process, with an average of one per decade, and volcano eruptions are an independent Poisson point process, with an average of two per decade. The $V$ be length of time (in decades) until the first volcano eruption and $E$ the length of time (in decades) until the first earthquake. Compute the following:

- $\mathbb{E}[E^2]$ and $\text{Cov}[E, V]$.
- The expected number of calendar years, in the next decade (ten calendar years), that have no earthquakes and no volcano eruptions.
- The probability density function of $\min\{E, V\}$. 

\[\]
Probability of no earthquake or eruption in first year is $e^{-2}$. Same for any year by memoryless property. Expected number of quake/eruption-free years is $10e^{-3} \approx 7.4$.

Probability density function of min $\{E, V\}$ is $3e^{-x}$ for $x \geq 0$, and 0 for $x < 0$.

- $E[E^2] = 2$ and $\text{Cov}[E, V] = 0$. 

Poisson point processes answers

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Poisson point processes answers

- $E[E^2] = 2$ and $\text{Cov}[E, V] = 0$.
- Probability of no earthquake or eruption in first year is $e^{-(2+1)\frac{1}{10}} = e^{-0.3}$ (see next part). Same for any year by memoryless property. Expected number of quake/eruption-free years is $10e^{-0.3} \approx 7.4$. 

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Poisson point processes

- $E[E^2] = 2$ and $\text{Cov}[E, V] = 0$.

- Probability of no earthquake or eruption in first year is $e^{-\frac{(2+1)}{10}} = e^{-0.3}$ (see next part). Same for any year by memoryless property. Expected number of quake/eruption-free years is $10e^{-0.3} \approx 7.4$.

- Probability density function of $\min\{E, V\}$ is $3e^{-(2+1)x}$ for $x \geq 0$, and 0 for $x < 0$. 

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