18.650. Statistics for Applications  
Fall 2016. Problem Set 5

Due Friday, Oct. 14 at 12 noon

Problem 1  Hypotheses testing and confidence intervals

Consider a sample $X_1, \ldots, X_n \overset{i.i.d.}{\sim} Poiss(\lambda)$, for some unknown $\lambda > 0$.

1. Based on the sample, propose a confidence interval for $\lambda$ with asymptotic level $1 - \alpha$, for some fixed $\alpha \in (0, 1)$ (prove your answer). Denote by $I$ your confidence interval.

2. Consider the following hypotheses, where $\lambda_0$ is a given positive number:

   $$H_0 : "\lambda = \lambda_0" \text{ vs. } H_1 : "\lambda \neq \lambda_0".$$  

   Using the confidence interval $I$, propose a test with asymptotic level $\alpha$ for these hypotheses.

Problem 2  Comparing two means

Consider two measuring instruments that are used to measure the intensity of some electromagnetic waves. An engineer wants to check if both instruments are calibrated identically, i.e., if they will produce identical measurements for identical waves. To do so, the engineer does $n_1$ independent measurements of the intensity of a given wave using the first instrument, and $n_2$ measurements on the same wave using the second instrument. The integers $n_1$ and $n_2$ may not be equal because, for instance, one instrument may be more costly than the other one, or may produce measurements more slowly. The measurements are denoted by $X_1, \ldots, X_{n_1}$ for the first instrument and by $Y_1, \ldots, Y_{n_2}$ for the second one. Intrinsic defects of the instruments will lead to measurement errors, and it is reasonable to assume that $X_1, \ldots, X_{n_1}$ are iid Gaussian and so are $Y_1, \ldots, Y_{n_2}$. If the two instruments are identically calibrated, the $X_i$'s and the $Y_i$'s should have the same expectation but may not have the same variance, since the two instruments may not have the same precision.

Hence, we assume that $X_1, \ldots, X_{n_1} \overset{i.i.d.}{\sim} \mathcal{N}(\mu_1, \sigma_1^2)$ and $Y_1, \ldots, Y_{n_2} \overset{i.i.d.}{\sim} \mathcal{N}(\mu_2, \sigma_2^2)$, where $\mu_1, \mu_2 \in \mathbb{R}$ and $\sigma_1^2, \sigma_2^2 > 0$, and that the two samples are independent of each other. We want to test whether $\mu_1 = \mu_2$.

1. Recall the expression of the maximum likelihood estimators for $(\mu_1, \sigma_1^2)$ and for $(\mu_2, \sigma_2^2)$. Denote these estimators by $(\hat{\mu}_1, \hat{\sigma}_1^2)$ and $(\hat{\mu}_2, \hat{\sigma}_2^2)$.
2. Recall the distribution of $\frac{n_1\sigma_1^2}{\sigma_1^2}$ and of $\frac{n_2\sigma_2^2}{\sigma_2^2}$.

3. What is the distribution of $\frac{n_1\hat{\sigma}_1^2}{\sigma_1^2} + \frac{n_2\hat{\sigma}_2^2}{\sigma_2^2}$?

4. Let $\Delta = \hat{\mu}_1 - \hat{\mu}_2$. What is the distribution of $\Delta$?

5. Consider the following hypotheses:

$$H_0 : "\mu_1 = \mu_2" \text{ vs. } H_0 : "\mu_1 = \mu_2".$$  

Here and in the next question we assume that $\sigma_1^2 = \sigma_2^2$. Based on the previous questions, propose a test with non-asymptotic level $\alpha \in (0,1)$ for $H_0$ against $H_1$.

6. Assume that 10 measurements have been done for both machines. The first instrument measured 8.43 in average with sample variance 0.22 and the second instrument measured 8.07 with sample variance 0.17. Can you conclude that the calibrations of the two machines are significantly identical at level 5%? What is, approximately, the p-value of your test?

**Problem 3** Implicit hypotheses testing

Based on a sample of i.i.d. Gaussian random variables $X_1, \ldots, X_n$ with mean $\mu$ and variance $\sigma^2$, propose a test with asymptotic level 5% for the hypotheses

$$H_0 : "\mu > \sigma" \text{ vs. } H_0 : "\mu \leq \sigma."$$

What is the p-value of your test if the sample has size $n = 100$, the sample average is 2.41 and the sample variance is 5.20? If the sample size is $n = 100$, the sample average is 3.28 and the sample variance is 15.95? In the latter case, do you reject $H_0$ at level 5%? At level 10%?