Chapter 1: Introduction
Goals

Goals:

▶ To give you a solid introduction to the mathematical theory behind statistical methods;

▶ To provide theoretical guarantees for the statistical methods that you may use for certain applications.

At the end of this class, you will be able to

1. From a real-life situation, formulate a statistical problem in mathematical terms

2. Select appropriate statistical methods for your problem

3. Understand the implications and limitations of various methods
Instructors

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  Associate Prof. of Applied Mathematics; IDSS; MIT Center for Statistics and Data Science.

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  Instructor in Applied Mathematics; IDSS; MIT Center for Statistics and Data Science.
Logistics

- Lectures: Tuesdays & Thursdays 1:00 -2:30am
- **Optional Recitation**: TBD.
- Homework: weekly. Total 11, 10 best kept (30%).
- Midterm: Nov. 8, in class, 1 hours and 20 minutes (30 %). Closed books closed notes. Cheatsheet.
- Final: TBD, 2 hours (40%). Open books, open notes.
Prerequisites: Probability (18.600 or 6.041), Calculus 2, notions of linear algebra (matrix, vector, multiplication, orthogonality,...)

Reading: There is no required textbook

Slides are posted on course website
https://ocw.mit.edu/courses/mathematics/18-650-statistics-for-applications-fall-2016/lecture-slides

Videolectures: Each lecture is recorded and posted online. Attendance is still recommended.
Why statistics?
Not only in the press

**Hydrology**  Netherlands, 10th century, building dams and dykes Should be high enough for most floods Should not be too expensive (high)

**Insurance**  Given your driving record, car information, coverage. What is a fair premium?

**Clinical trials**  A drug is tested on 100 patients; 56 were cured and 44 showed no improvement. Is the drug effective?
Randomness

What is common to all these examples?
Randomness

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RANDOMNESS
Randomness

What is common to all these examples?

RANDOMNESS

Associated questions:

- Notion of average ("fair premium", ...)
- Quantifying chance ("most of the floods", ...)
- Significance, variability, ...
Probability

- Probability studies randomness (hence the prerequisite)
- Sometimes, the physical process is completely known: dice, cards, roulette, fair coins, ...

Examples

Rolling 1 die:
- Alice gets $1 if # of dots ≤ 3
- Bob gets $2 if # of dots ≤ 2
Who do you want to be: Alice or Bob?

Rolling 2 dice:
- Choose a number between 2 and 12
- Win $100 if you chose the sum of the 2 dice
Which number do you choose?

Well known random process from physics: 1/6 chance of each side, dice are independent. We can deduce the probability of outcomes, and expected $ amounts. This is probability.
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Statistics and modeling

- How about more complicated processes? Need to estimate parameters from data. This is statistics
- Sometimes real randomness (random student, biased coin, measurement error, …)
- Sometimes deterministic but too complex phenomenon: statistical modeling
  - Complicated process “≡” Simple process + random noise
- (good) Modeling consists in choosing (plausible) simple process and noise distribution.
Statistics vs. probability

**Probability**  Previous studies showed that the drug was 80% effective. Then we can anticipate that for a study on 100 patients, in average 80 will be cured and at least 65 will be cured with 99.99% chances.

**Statistics**  Observe that 78/100 patients were cured. We (will be able to) conclude that we are 95% confident that for other studies the drug will be effective on between 69.88% and 86.11% of patients.
What this course is about

- Understand **mathematics** behind statistical methods
- Justify quantitative statements given modeling assumptions
- Describe interesting mathematics arising in statistics
- Provide a math toolbox to extend to other models.

What this course is **not** about

- Statistical thinking/modeling (applied stats, e.g. IDS.012)
- Implementation (computational stats, e.g. IDS.012)
- Laundry list of methods (boring stats, e.g. AP stats)
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Let’s do some statistics
“A neonatal right-side preference makes a surprising romantic reappearance later in life.”

- Let $p$ denote the proportion of couples that turn their head to the right when kissing.
- Let us design a statistical experiment and analyze its outcome.
- Observe $n$ kissing couples times and collect the value of each outcome (say 1 for RIGHT and 0 for LEFT);
- Estimate $p$ with the proportion $\hat{p}$ of RIGHT.
- Study: “Human behaviour: Adult persistence of head-turning asymmetry” (Nature, 2003): $n = 124$, 80 to the right so

$$\hat{p} = \frac{80}{124} = 64.5\%$$
Heuristics (2)

Back to the data:

- 64.5% is much larger than 50% so there seems to be a preference for turning right.
- What if our data was RIGHT, RIGHT, LEFT \((n = 3)\). That’s 66.7% to the right. Even better?
- Intuitively, we need a large enough sample size \(n\) to make a call. How large?

We need **mathematical modeling** to understand the accuracy of this procedure?
Heuristics (3)

Formally, this procedure consists of doing the following:

- For $i = 1, \ldots, n$, define $R_i = 1$ if the $i$th couple turns to the right RIGHT, $R_i = 0$ otherwise.
- The estimator of $p$ is the sample average

$$\hat{p} = \bar{R}_n = \frac{1}{n} \sum_{i=1}^{n} R_i.$$ 

What is the accuracy of this estimator?

In order to answer this question, we propose a statistical model that describes/approximates well the experiment.
Coming up with a model consists of making assumptions on the observations $R_i, i = 1, \ldots, n$ in order to draw statistical conclusions. Here are the assumptions we make:

1. Each $R_i$ is a random variable.

2. Each of the r.v. $R_i$ is Bernoulli with parameter $p$.

3. $R_1, \ldots, R_n$ are mutually independent.
Heuristics (5)

Let us discuss these assumptions.

1. Randomness is a way of modeling lack of information; with perfect information about the conditions of kissing (including what goes in the kissers’ mind), physics or sociology would allow us to predict the outcome.

2. Hence, the $R_i$’s are necessarily Bernoulli r.v. since $R_i \in \{0, 1\}$. They could still have a different parameter $R_i \sim \text{Ber}(p_i)$ for each couple but we don’t have enough information with the data estimate the $p_i$’s accurately. So we simply assume that our observations come from the same process: $p_i = p$ for all $i$

3. Independence is reasonable (people were observed at different locations and different times).
Two important tools: LLN & CLT

Let $X, X_1, X_2, \ldots, X_n$ be i.i.d. r.v., $\mu = \mathbb{E}[X]$ and $\sigma^2 = \mathbb{V}[X]$.

- Laws of large numbers (weak and strong):

  \[
  \bar{X}_n := \frac{1}{n} \sum_{i=1}^{n} X_i \xrightarrow{\mathbb{P}, \text{a.s.}} \mu.
  \]

- Central limit theorem:

  \[
  \sqrt{n} \frac{\bar{X}_n - \mu}{\sigma} \xrightarrow{(d)} \mathcal{N}(0, 1).
  \]

  (Equivalently, $\sqrt{n} (\bar{X}_n - \mu) \xrightarrow{(d)} \mathcal{N}(0, \sigma^2)$.)
Consequences (1)

- The LLN’s tell us that

\[ \bar{R}_n \xrightarrow{\mathbb{P}, \text{a.s.}} p. \]

- Hence, when the size $n$ of the experiment becomes large, $\bar{R}_n$ is a good (say "consistent") estimator of $p$.

- The CLT refines this by quantifying how good this estimate is.
Consequences (2)

$$
\Phi(x): \text{cdf of } \mathcal{N}(0, 1);
$$

$$
\Phi_n(x): \text{cdf of } \sqrt{n} \frac{\bar{R}_n - p}{\sqrt{p(1 - p)}}.
$$

CLT: \( \Phi_n(x) \approx \Phi(x) \) when \( n \) becomes large. Hence, for all \( x > 0 \),

$$
\mathbb{P} [ | \bar{R}_n - p | \geq x ] \approx 2 \left( 1 - \Phi \left( \frac{x\sqrt{n}}{\sqrt{p(1 - p)}} \right) \right).
$$
Consequences (3)

Consequences:

- Approximation on how $\bar{R}_n$ concentrates around $p$;

- For a fixed $\alpha \in (0, 1)$, if $q_{\alpha/2}$ is the $(1 - \alpha/2)$-quantile of $\mathcal{N}(0, 1)$, then with probability $\approx 1 - \alpha$ (if $n$ is large enough!),

$$
\bar{R}_n \in \left[ p - \frac{q_{\alpha/2} \sqrt{p(1-p)}}{\sqrt{n}}, p + \frac{q_{\alpha/2} \sqrt{p(1-p)}}{\sqrt{n}} \right].
$$
Consequences (4)

- Note that no matter the (unknown) value of $p$,
  \[ p(1 - p) \leq 1/4. \]

- Hence, roughly with probability at least $1 - \alpha$,
  \[ \bar{R}_n \in \left[ p - \frac{q\alpha/2}{2\sqrt{n}}, p + \frac{q\alpha/2}{2\sqrt{n}} \right]. \]

- In other words, when $n$ becomes large, the interval
  \[ \left[ \bar{R}_n - \frac{q\alpha/2}{2\sqrt{n}}, \bar{R}_n + \frac{q\alpha/2}{2\sqrt{n}} \right] \]
  contains $p$ with probability $\geq 1 - \alpha$.

- This interval is called an asymptotic confidence interval for $p$.

- In the kiss example, we get
  \[ \left[ 0.645 \pm \frac{1.96}{2\sqrt{124}} \right] = [0.56, 0.73] \]

If the extreme ($n = 3$ case) we would have $[0.10, 1.23]$ but CLT is not valid! Actually we can make exact computations!
Another useful tool: Hoeffding’s inequality

What if \( n \) is not so large?

Hoeffding’s inequality (i.i.d. case)

Let \( n \) be a positive integer and \( X, X_1, \ldots, X_n \) be i.i.d. r.v. such that \( X \in [a, b] \) a.s. (\( a < b \) are given numbers). Let \( \mu = \mathbb{E}[X] \).

Then, for all \( \varepsilon > 0 \),

\[
\mathbb{P} \left[ \left| \bar{X}_n - \mu \right| \geq \varepsilon \right] \leq 2e^{-\frac{2n\varepsilon^2}{(b-a)^2}}.
\]

Consequence:

- For \( \alpha \in (0, 1) \), with probability \( \geq 1 - \alpha \),

\[
\bar{R}_n - \sqrt{\frac{\log(2/\alpha)}{2n}} \quad p \quad \bar{R}_n + \sqrt{\frac{\log(2/\alpha)}{2n}}.
\]

- This holds even for small sample sizes \( n \).
Review of different types of convergence (1)

Let \((T_n)_{n \geq 1}\) a sequence of r.v. and \(T\) a r.v. (\(T\) may be deterministic).

- **Almost surely (a.s.) convergence:**
  \[
  T_n \xrightarrow{\text{a.s.}} T \quad \text{iff} \quad \Pr \left[ \left\{ \omega : T_n(\omega) \xrightarrow{n \to \infty} T(\omega) \right\} \right] = 1.
  \]

- **Convergence in probability:**
  \[
  T_n \xrightarrow{\Pr} T \quad \text{iff} \quad \Pr \left[ |T_n - T| \geq \varepsilon \right] \xrightarrow{n \to \infty} 0, \quad \forall \varepsilon > 0.
  \]
Review of different types of convergence (2)

- Convergence in $L^p$ ($p \geq 1$):

\[ T_n \xrightarrow{L^p} T \quad \text{iff} \quad \operatorname{IE} [ |T_n - T|^p ] \xrightarrow{n \to \infty} 0. \]

- Convergence in distribution:

\[ T_n \xrightarrow{(d)} T \quad \text{iff} \quad \operatorname{IP} [ T_n \ x] \xrightarrow{n \to \infty} \operatorname{IP} [ T \ x], \]

for all $x \in \mathbb{R}$ at which the cdf of $T$ is continuous.

**Remark**

These definitions extend to random vectors (i.e., random variables in $\mathbb{R}^d$ for some $d \geq 2$).
Important characterizations of convergence in distribution

The following propositions are equivalent:

(i) \( T_n \xrightarrow{d}{\frac{(d)}{n\to\infty}} T; \)

(ii) \( \mathbb{E}[f(T_n)] \xrightarrow{n\to\infty} \mathbb{E}[f(T)], \) for all continuous and bounded function \( f; \)

(iii) \( \mathbb{E} \left[ e^{ixT_n} \right] \xrightarrow{n\to\infty} \mathbb{E} \left[ e^{ixT} \right], \) for all \( x \in \mathbb{R}. \)
Important properties

▶ If \((T_n)_{n \geq 1}\) converges a.s., then it also converges in probability, and the two limits are equal a.s.

▶ If \((T_n)_{n \geq 1}\) converges in \(L^p\), then it also converges in \(L^q\) for all \(q \geq p\) and in probability, and the limits are equal a.s.

▶ If \((T_n)_{n \geq 1}\) converges in probability, then it also converges in distribution

▶ If \(f\) is a continuous function:

\[
T_n \xrightarrow{a.s./\mathbb{P}/(d)} T \quad \Rightarrow \quad f(T_n) \xrightarrow{a.s./\mathbb{P}/(d)} f(T).
\]
Review of different types of convergence (6)

Limits and operations

One can add, multiply, ... limits almost surely and in probability. If
\[ U_n \xrightarrow{a.s./\mathbb{P}} U \quad \text{and} \quad V_n \xrightarrow{a.s./\mathbb{P}} V, \]
then:

- \[ U_n + V_n \xrightarrow{n \to \infty} U + V, \]
- \[ U_n V_n \xrightarrow{n \to \infty} UV, \]
- If in addition, \( V \neq 0 \) a.s., then \[ \frac{U_n}{V_n} \xrightarrow{n \to \infty} \frac{U}{V}. \]

⚠️ In general, these rules do not apply to convergence in distribution unless the pair \((U_n, V_n)\) converges in distribution to \((U, V)\).
Another example (1)

- You observe the times between arrivals of the T at Kendall: $T_1, \ldots, T_n$.

- You assume that these times are:
  - Mutually independent
  - Exponential random variables with common parameter $\lambda > 0$.

- You want to estimate the value of $\lambda$, based on the observed arrival times.
Another example (2)

Discussion of the assumptions:

- Mutual independence of $T_1, \ldots, T_n$: plausible but not completely justified (often the case with independence).

- $T_1, \ldots, T_n$ are exponential r.v.: **lack of memory** of the exponential distribution:

$$\mathbb{P}[T_1 > t + s | T_1 > t] = \mathbb{P}[T_1 > s], \quad \forall s, t \geq 0.$$ 

Also, $T_i > 0$ almost surely!

- The exponential distributions of $T_1, \ldots, T_n$ have the same parameter: in average all the same inter-arrival time. True only for limited period (rush hour $\neq$ 11pm).
Another example (3)

- **Density of** $T_1$:
  \[ f(t) = \lambda e^{-\lambda t}, \quad \forall t \geq 0. \]

- \( \mathbb{E}[T_1] = \frac{1}{\lambda}. \)

- **Hence**, a natural estimate of \( \frac{1}{\lambda} \) is
  \[ \bar{T}_n := \frac{1}{n} \sum_{i=1}^{n} T_i. \]

- **A natural estimator of** $\lambda$ is
  \[ \hat{\lambda} := \frac{1}{\bar{T}_n}. \]
Another example (4)

- By the LLN’s,
  \[ \bar{T}_n \xrightarrow{a.s./\mathbb{P}} \frac{1}{\lambda} \quad \text{as } n \to \infty \]

- Hence,
  \[ \hat{\lambda} \xrightarrow{a.s./\mathbb{P}} \lambda. \quad \text{as } n \to \infty \]

- By the CLT,
  \[ \sqrt{n} \left( \bar{T}_n - \frac{1}{\lambda} \right) \xrightarrow{(d)} \mathcal{N}(0, \lambda^{-2}). \quad \text{as } n \to \infty \]

- How does the CLT transfer to \( \hat{\lambda} \)? How to find an asymptotic confidence interval for \( \lambda \)?
The Delta method

Let \((Z_n)_{n\geq 1}\) sequence of r.v. that satisfies

\[
\sqrt{n}(Z_n - \theta) \xrightarrow{d}_{n \to \infty} N(0, \sigma^2),
\]

for some \(\theta \in \mathbb{R}\) and \(\sigma^2 > 0\) (the sequence \((Z_n)_{n\geq 1}\) is said to be asymptotically normal around \(\theta\)).

Let \(g : \mathbb{R} \to \mathbb{R}\) be continuously differentiable at the point \(\theta\). Then,

- \((g(Z_n))_{n\geq 1}\) is also asymptotically normal;
- More precisely,

\[
\sqrt{n} \left( g(Z_n) - g(\theta) \right) \xrightarrow{d}_{n \to \infty} N(0, g'(\theta)^2 \sigma^2).
\]
Consequence of the Delta method (1)

- \( \sqrt{n} \left( \hat{\lambda} - \lambda \right) \xrightarrow{n \to \infty} \mathcal{N}(0, \lambda^2). \)

- Hence, for \( \alpha \in (0, 1) \) and when \( n \) is large enough,
  \[ |\hat{\lambda} - \lambda| \xrightarrow{n \to \infty} \frac{q_{\alpha/2}\lambda}{\sqrt{n}}. \]

- Can \( \left[ \hat{\lambda} - \frac{q_{\alpha/2}\lambda}{\sqrt{n}}, \hat{\lambda} + \frac{q_{\alpha/2}\lambda}{\sqrt{n}} \right] \) be used as an asymptotic confidence interval for \( \lambda \)?

- **No!** It depends on \( \lambda \)...
Consequence of the Delta method (2)

Two ways to overcome this issue:

▶ In this case, we can solve for $\lambda$:

$$|\hat{\lambda} - \lambda| \frac{q_{\alpha/2}\lambda}{\sqrt{n}} \iff \lambda \left(1 - \frac{q_{\alpha/2}}{\sqrt{n}}\right) \hat{\lambda} \lambda \left(1 + \frac{q_{\alpha/2}}{\sqrt{n}}\right)$$

$$\iff \hat{\lambda} \left(1 + \frac{q_{\alpha/2}}{\sqrt{n}}\right)^{-1} \lambda \lambda \left(1 - \frac{q_{\alpha/2}}{\sqrt{n}}\right)^{-1}.$$

Hence, $\left[\hat{\lambda} \left(1 + \frac{q_{\alpha/2}}{\sqrt{n}}\right)^{-1}, \lambda \left(1 - \frac{q_{\alpha/2}}{\sqrt{n}}\right)^{-1}\right]$ is an asymptotic confidence interval for $\lambda$.

▶ A systematic way: Slutsky's theorem.
Slutsky’s theorem

Let \((X_n), (Y_n)\) be two sequences of r.v., such that:

(i) \(X_n \xrightarrow{(d)_{n \to \infty}} X;\)

(ii) \(Y_n \xrightarrow{P_{n \to \infty}} c,\)

where \(X\) is a r.v. and \(c\) is a given real number. Then,

\[ (X_n, Y_n) \xrightarrow{(d)_{n \to \infty}} (X, c). \]

In particular,

\[ X_n + Y_n \xrightarrow{(d)_{n \to \infty}} X + c, \]

\[ X_nY_n \xrightarrow{(d)_{n \to \infty}} cX, \]

\[ \ldots \]
Consequence of Slutsky’s theorem (1)

- Thanks to the Delta method, we know that

\[
\sqrt{n} \frac{\hat{\lambda} - \lambda}{\lambda} \xrightarrow{(d)\, n \to \infty} \mathcal{N}(0, 1).
\]

- By the weak LLN,

\[
\hat{\lambda} \xrightarrow{\text{IP} \, n \to \infty} \lambda.
\]

- Hence, by Slutsky’s theorem,

\[
\sqrt{n} \frac{\hat{\lambda} - \lambda}{\hat{\lambda}} \xrightarrow{(d)\, n \to \infty} \mathcal{N}(0, 1).
\]

- Another asymptotic confidence interval for \( \lambda \) is

\[
\left[ \hat{\lambda} - \frac{q_{\alpha/2}\hat{\lambda}}{\sqrt{n}}, \hat{\lambda} + \frac{q_{\alpha/2}\hat{\lambda}}{\sqrt{n}} \right].
\]
Remark:

- In the first example (kisses), we used a problem dependent trick: “\( p(1 - p) \ 1/4 \).”

- We could have used Slutsky’s theorem and get the asymptotic confidence interval

\[
\left[ \bar{R}_n - \frac{q_{\alpha/2} \sqrt{\bar{R}_n (1 - \bar{R}_n)}}{\sqrt{n}}, \bar{R}_n + \frac{q_{\alpha/2} \sqrt{\bar{R}_n (1 - \bar{R}_n)}}{\sqrt{n}} \right].
\]