One-sided inverses

These notes are a small extension of the material on pages 53–55 of the text.

Definition 1. Suppose $V$ and $W$ are vector spaces over a field $F$, and $T \in \mathcal{L}(V, W)$. A left inverse for $T$ is a linear map $S \in \mathcal{L}(W, V)$ with the property that $ST = I_V$ (the identity map on $V$). That is, we require
\[ ST(v) = v \quad (\text{all } v \in V). \]

A right inverse for $T$ is a linear map $S' \in \mathcal{L}(W, V)$ with the property that $T S' = I_W$ (the identity map on $W$). That is, we require
\[ TS'(w) = w \quad (\text{all } w \in W). \]

What are these things good for? I’ve said that one of the most basic problems in linear algebra is solving an equation like
\[ T x = c \quad (\text{QUESTION}) \]
(with $c \in W$ specified); you are to find the unknown $x \in V$. If $S$ is a left inverse of $T$, then we can apply $S$ to this equation and get
\[ x = I_V(x) = ST x = Sc. \quad (\text{LEFT}) \]

What this calculation proves is

Proposition 2. Suppose $S$ is a left inverse of $T$. Then the only possible solution of (QUESTION) is $x = Sc$.

This does not say that $Sc$ really is a solution; just that it’s the only candidate for a solution. Sometimes that’s useful information.

On the other hand, suppose $S'$ is a right inverse of $T$. Then we can try $x = S'c$ and get
\[ Tx = T S' c = I_W c = c. \quad (\text{RIGHT}) \]

This calculation proves

Proposition 3. Suppose $S'$ is a right inverse of $T$. Then $x = S'c$ is a solution of (QUESTION).

This time the ambiguity is uniqueness: we have found one solution, but there may be others. Sometimes that’s all we need.

Example. Suppose $V = W = \mathcal{P}(\mathbb{R})$ (polynomials), and $D = \frac{d}{dx}$. We would like to “undo” differentiation, so we integrate:
\[ (Jp)(x) = \int_0^x p(t) \, dt. \]

The fundamental theorem of calculus says that the derivative of this integral is $p$; that is, $DJ = I_p$. So $J$ is a right inverse of $D$; it provides a solution (not the only one!) of the differential equation $\frac{4q}{dx} = p$. If we try things in the other direction, there is a problem:
\[ JD(p) = \int_0^x p'(t) \, dt = p(x) - p(0). \]

That is, $JD$ sends $p$ to $p - p(0)$, which is not the same as $p$. So $J$ is not a left inverse to $D$; since $D$ has a nonzero null space, we’ll see that no left inverse can exist.
Theorem 4. Suppose $V$ and $W$ are finite-dimensional, and that $T \in \mathcal{L}(V,W)$.
1) The operator $T$ has a left inverse if and only if $\text{Null}(T) = 0$.
2) If $S$ is a left inverse of $T$, then $\text{Null}(S)$ is a complement to $\text{Range}(T)$ in the sense of Proposition 2.13 in the text:

$$W = \text{Range}(T) \oplus \text{Null}(S).$$

3) Assuming that $\text{Null}(T) = 0$, there is a one-to-correspondence between left inverses of $T$ and subspaces of $W$ complementary to $\text{Range}(T)$.
4) The operator $T$ has a right inverse if and only if $\text{Range}(T) = W$.
5) If $S'$ is a right inverse of $T$, then $\text{Range}(S')$ is a complement to $\text{Null}(T)$ in the sense of Proposition 2.13 in the text:

$$V = \text{Null}(T) \oplus \text{Range}(S').$$

6) Assuming that $\text{Range}(T) = W$, there is a one-to-correspondence between right inverses of $T$ and subspaces of $V$ complementary to $\text{Null}(T)$.
7) If $T$ has both a left and a right inverse, then the left and right inverses are unique and equal to each other. That is, there is a unique linear map $S \in \mathcal{L}(W,V)$ characterized by either of the two properties $ST = I_V$ or $TS = I_W$. If it has one of these properties, then it automatically has the other.

Proof. For (1), suppose first that a left inverse exists. According to Proposition 2, the equation $Tx = 0$ has at most one solution, namely $x = S0 = 0$. That says precisely that $\text{Null}(T) = 0$. Conversely, suppose $\text{Null}(T) = 0$. Choose a basis $(v_1, \ldots, v_n)$ of $V$. By the proof of the rank plus nullity theorem, $(Tv_1, \ldots, Tv_n)$ is a basis of $\text{Range}(T)$; so in particular it is a linearly independent set in $W$. We may therefore extend it to a basis

$$(Tv_1, \ldots, Tv_n, w_1, \ldots w_p)$$

of $W$.

To define a linear map $S$ from $W$ to $V$, we need to pick the images of these $n + p$ basis vectors; we are allowed to pick any vectors in $V$. If $S$ is going to be a left inverse of $T$, we are forced to choose

$$S(Tv_i) = v_i;$$

the choices of $Sw_j$ can be arbitrary. Since we have then arranged for the equation $STv = v$ to be true for all elements of a basis of $V$, it must be true for all of $V$. Therefore $S$ is a left inverse of $T$.

For (2), suppose $ST = I_V$; we need to prove the direct sum decomposition shown. So suppose $w \in W$. Define $v = Sw$ and $r = Tv = TSw \in W$. Then $r \in \text{Range}(T)$, and

$$n = w - r = w - TSw$$

satisfies

$$Sn = Sw - STSw = Sw - I_V Sw = Sw - Sw = 0;$$
so $n \in \text{Null}(S)$. We have therefore written $w = r + n$ as the sum of an element of $\text{Range}(T)$ and of $\text{Null}(S)$. To prove that the sum is direct, we must show that $\text{Null}(S) \cap \text{Range}(T) = 0$. So suppose $Tv$ (in $\text{Range}(T)$) is also in $\text{Null}(S)$. Then

$$v = STv = 0$$

(since $Tv \in \text{Null}(S)$) so also $Tv = 0$, as we wished to show.

For (3), we have seen that any left inverse gives a direct sum decomposition of $W$. Conversely, suppose that $W = \text{Range}(T) \oplus N$ is a direct sum decomposition. Define a linear map $S$ from $W$ to $V$ by

$$S(Tv + n) = v \quad (v \in V, n \in N).$$

This formula makes sense because there is only one $v$ with image $Tv$ (by $\text{Null}(T) = 0$); it defines $S$ on all of $W$ by the direct sum hypothesis. This construction makes a left inverse $S$ with $\text{Null}(S) = N$, and in fact it is the only way to make a left inverse with this null space.

Parts (4)–(6) are proved in exactly the same way.

For (7), if the left and right inverses exist, then $\text{Null}(T) = 0$ and $\text{Range}(T) = W$. So the only possible complement to $\text{Range}(T)$ is 0, so the left inverse $S$ is unique by (3); and the only possible complement to $\text{Null}(T)$ is $V$, so the right inverse is unique by (6). To see that they are equal, apply $S'$ on the right to the equation $ST = I_V$; we get

$$S' = I_V S' = ST S' = S I_W = S,$$

so the left and right inverses are equal.