Row Rank = Column Rank

This is in remorse for the mess I made at the end of class on Oct 1.

The column rank of an \( m \times n \) matrix \( A \) is the dimension of the subspace of \( F^m \) spanned by the columns of \( A \). Similarly, the row rank is the dimension of the subspace of the space \( F^n \) of row vectors spanned by the rows of \( A \).

**Theorem.** The row rank and the column rank of a matrix \( A \) are equal.

**proof.** We have seen that there exist an invertible \( m \times m \) matrix \( Q \) and an invertible \( n \times n \) matrix \( P \) such that \( A_1 = Q^{-1}AP \) has the block form

\[
A_1 = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}
\]

where \( I \) is an \( r \times r \) identity matrix for some \( r \), and the rest of the matrix is zero. For this matrix, it is obvious that row rank = column rank = \( r \). The strategy is to reduce an arbitrary matrix to this form.

We can write \( Q^{-1} = E_k \cdots E_2 E_1 \) and \( P = E'_1 E'_2 \cdots E'_r \) for some elementary \( m \times m \) matrices \( E_i \) and \( n \times n \) matrices \( E'_j \). So \( A_1 \) is obtained from \( A \) by a sequence of row and column operations. (It doesn’t matter whether one does the row operations before the column operations, or mixes them together: The associative law for matrix multiplication shows that \( E(\text{AE}') = (\text{EA})E' \), i.e., that row operations commute with column operations.)

This being so, it suffices to show that the row ranks and column ranks of a matrix \( A \) are equal to those of a matrix of the form \( EA \), and also to those of a matrix of the form \( \text{AE}' \). We’ll treat the case of a row operation \( EA \). The column operation \( \text{AE}' \) can be analyzed in a similar way, or one can use the transpose to change row operations to column operations.

We denote the matrix \( EA \) by \( A' \). Let the columns of \( A \) be \( C_1, ..., C_n \) and let those of \( A' \) be \( C'_1, ..., C'_n \).

Then \( C'_j = EC_j \). Therefore any linear relation among the columns of \( A \) gives us a linear relation among the columns of \( A' \): If \( C_1 x_1 + \cdots + C_n x_n = 0 \) then

\[
E(C_1 x_1 + \cdots + C_n x_n) = C'_1 x_1 + \cdots + C'_n x_n = 0.
\]

So if \( j_1, ..., j_r \) are distinct indices between 1 and \( n \), and if the set \( \{C'_{j_1}, ..., C'_{j_r}\} \) is independent, the set \( \{C_{j_1}, ..., C_{j_r}\} \) must also be independent. This shows that

\[
\text{column rank}(A') \leq \text{column rank}(A).
\]

Because the inverse of an elementary matrix is elementary and \( A = E^{-1}A' \), we can also conclude that \( \text{column rank}(A) \leq \text{column rank}(A') \). The column ranks of the two matrices are equal.

Next, let the rows of \( A \) be \( R_1, ..., R_n \) and let those of \( A' \) be \( R'_1, ..., R'_n \), and let’s suppose that \( E \) is an elementary matrix of the first type, that adds \( a \cdot \text{row } k \) to \( \text{row } i \). So \( R'_j = R_j \) for \( j \neq i \) and \( R'_i = R_i + aR_k \). Then any linear combination of the rows \( R'_j \) is also a linear combination of the rows \( R_j \). Therefore \( \text{Span}\{R'_j\} \subseteq \text{Span}\{R_j\} \), and so \( \text{row rank}(A') \leq \text{row rank}(A) \). And because the inverse of \( E \) is elementary, we obtain the other inequality. Elementary matrices of the other types are treated easily, so the row ranks of the two matrices are equal. \( \square \)
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