18.702 Homework for Wednesday, February 16, corrected

First version omitted the very important requirement \( d(\gamma_0) = 1 \) from the definition of a harmonic graph.


2. Chapter 10, Exercise 6.9 (identify simple groups from character table).

3. A connected finite graph is called harmonic if it has the extra structure of
   1) a positive integer weight \( d(\gamma) \) labeling each vertex \( \gamma \); and
   2) a special vertex called \( \gamma_0 \), with \( d(\gamma_0) = 1 \)
   subject to the following requirement: for every vertex \( \gamma \),
   \[
   2d(\gamma) = \sum_{\text{edges } \{\gamma, \gamma'\}} d(\gamma').
   \]
   That is, the label of each vertex is half the sum of the labels of adjacent vertices. The problem is
to classify all possible harmonic graphs. (Here is one family of examples, to get you started. Fix
an integer \( N \geq 3 \). Let the vertices be the vertices of a regular \( N \)-gon, and the edges the usual
edges. Label each vertex with a \( 1 \), and pick one vertex to call \( \gamma_0 \). Since each vertex has exactly
two adjacent vertices, it’s easy to see that this is a harmonic graph.)

4. Suppose \( G \) is a finite subgroup of \( SU_2 \). Write \( (\tau, \mathbb{C}^2) \) for the defining representation of \( G \);
   the \( \tau \) stands for “tautological.” Now make a graph having one vertex \( \gamma \) for every (equivalence class
   of) irreducible representation \( (\gamma, V_{\gamma}) \) of \( G \). There is a distinguished vertex \( \gamma_0 \) corresponding to the
   trivial representation of \( G \).
   Join the vertex \( \gamma' \) to \( \gamma \) by \( e(\gamma', \gamma) \) (directed) edges if the irreducible representation \( \gamma' \) appears
   \( e(\gamma', \gamma) \) times in the representation \( \text{Hom}(\tau, \gamma) \).
   a) Show that \( e(\gamma', \gamma) \) is equal to
   \[
   \langle \chi_{\gamma'}^\tau, \chi_{\gamma} \rangle = |G|^{-1} \sum_{g \in G} \chi_{\gamma'}^\tau(g) \chi_{\gamma}(g).
   \]
   b) Show that \( e(\gamma', \gamma) = e(\gamma, \gamma') \); so we can think of the edges as undirected.
   c) Label each vertex \( \gamma \) by \( d(\gamma) = \text{dim} V_{\gamma} \). Show that the graph is harmonic. (The hard part is
   showing that the graph is connected; I haven’t really explained how you might prove that. So
   don’t worry if that’s the only part you can’t figure out.)
   d) In section 6.12 you studied finite subgroups of \( SO_3 \). In section 9.4 you studied a two-to-one
   map from \( SU_2 \) to \( SO_3 \). Every finite subgroup \( F \) of \( SO_3 \) has a preimage \( \tilde{F} \) in \( SU_2 \)
having twice as many elements, and called the binary something or other. For example, the preimage \( \tilde{T} \)
of the icosahedral group is called the binary icosahedral group; it has 120 elements. In your list
of harmonic graphs from problem 3, decide which ones correspond to the binary groups \( \tilde{T}, \tilde{O}, \)
and \( \tilde{I} \) (of orders 24, 48, and 120).