THEOREM (Refined Noether Normalization Lemma). Let \( k \) be a field, \( R \) a finitely generated \( k \)-algebra, and \( a_1 \subset \cdots \subset a_r \subsetneq R \) a chain of proper ideals. Then there exist algebraically independent elements \( t_1, \ldots, t_n \) of \( R \) such that

(a) \( R \) is module finite over \( k[t_1, \ldots, t_n] \);
(b) for \( 1 \leq i \leq r \), there is an \( h(i) \) such that \( a_i \cap k[t_1, \ldots, t_n] = (t_1, \ldots, t_{h(i)}) \).

PROOF (Cf. [Bourbaki, “Commutative Algebra,” Thm. 1, p. 344].) By hypothesis, \( R = S/b \) where \( S \) is a polynomial ring \( k[T_1, \ldots, T_m] \). Say \( a_i = b_i/b_0 \). Then it suffices to prove the assertion for \( S \) and \( b_0 \subset b_1 \subset \cdots \subset b_r \). Thus we may assume \( R \) is the polynomial algebra \( k[T_1, \ldots, T_m] \). The proof proceeds by induction on \( r \).

First, suppose \( r = 1 \) and \( a_1 \) is a principal ideal generated by a nonzero element \( t_1 \). Then \( t_1 \notin k \) because \( a_1 \neq R \). Write \( t_1 = \sum a_{(j)} T_1^{j_1} \cdots T_m^{j_m} \) where \( (j_1, \ldots, j_m) \in \mathbb{Z}_{\geq 0}^m \) and \( a_{(j)} \in k \) is nonzero. We are going to choose positive integers \( s_i \) for \( 2 \leq i \leq m \) such that \( T_1 \) is integral over \( R' := k[t_1, t_2, \ldots, t_m] \) where \( t_i := T_i - T_1^{s_i} \). Then clearly, (a) follows.

Note that \( T_1 \) satisfies the equation,

\[
t_1 - \sum a_{(j)} T_1^{j_1} (t_2 + T_1^{s_2}) \cdots (t_m + T_1^{s_m})^{j_m} = 0.
\]

Set \( e(j) := j_1 + s_2 j_2 + \cdots + s_m j_m \). Take \( s_i := \ell i \) where \( \ell \) is an integer greater than all of the \( j_i \). Then the \( e(j) \) are distinct. Let \( e(j') \) be largest \( e(j) \). Then the above equation can be written in the form

\[
a_{(j')} T_1^{e(j')} + \sum_{v < e(j')} Q_v T_1^v = 0
\]

where \( Q_v \in R' \), and hence, \( T_1 \) is integral over \( R' \). Thus (a) holds.

By the theory of transcendence bases [Artin, “Algebra,” Ch. 13, §8, pp. 525–527], the elements \( t_1, \ldots, t_m \) are algebraically independent. Let \( x \in a_1 \cap R' \). Then \( x = t_1 x' \) where \( x' \in R \cap k(t_1, \ldots, t_m) \). Furthermore, \( R \cap k(t_1, \ldots, t_m) = R' \) because \( R' \) is normal as it is a polynomial algebra. Hence \( a_1 \cap R' = t_1 R' \). Thus (b) holds in case \( r = 1 \) and \( a_1 \) is principal.

Second, suppose \( r = 1 \) and \( a_1 \) is arbitrary. If \( a_1 = 0 \), then we may take \( t_i := t_i \). Also assume \( a_1 \neq 0 \). The proof proceeds by induction on \( m \). The case \( m = 1 \) follows from the first case (but is simpler) because \( k[T_1] \) is a principal ring. Let \( t_1 \in a_1 \) be nonzero. By the first case, there exist elements \( u_2, \ldots, u_m \) such that \( t_1, u_2, \ldots, u_m \) are algebraically independent and satisfy (a) and (b) with respect to \( R \) and \( t_1 R \).

By induction, there exist elements \( t_2, \ldots, t_m \) satisfying (a) and (b) with respect to \( k[u_2, \ldots, u_m] \) and \( a_1 \cap k[u_2, \ldots, u_m] \).

Set \( R' := k[t_1, \ldots, t_m] \). Since \( R \) is module finite over \( k[t_1, u_2, \ldots, u_m] \) and the latter is module finite over \( R' \), the former is module finite over \( R' \). Hence (a) holds, and \( t_1, \ldots, t_m \) are algebraically independent. Moreover, by hypothesis,

\[
a_1 \cap k[t_2, \ldots, t_m] = (t_2, \ldots, t_h)
\]

for some \( h \leq m \). So \( a_1 \cap k[t_1, \ldots, t_m] \supset (t_1, \ldots, t_h) \).

Conversely, given \( x \in a_1 \cap R' \), write \( x = \sum_{i=0}^{n} Q_i t_i^j \) where \( Q_i \in k[t_1, \ldots, t_m] \). Since \( t_1 \in a_1 \), we have \( Q_0 \in a_1 \cap k[t_2, \ldots, t_m] \), so \( Q_{(0)} \in (t_2, \ldots, t_h) \). Hence \( x \in (t_1, \ldots, t_h) \). Thus \( a_1 \cap R' = (t_1, \ldots, t_h) \). Thus (b) holds for \( r = 1 \).
Finally, suppose the theorem holds for \( r - 1 \). Let \( u_1, \ldots, u_m \) be algebraically independent elements of \( R \) satisfying (a) and (b) for the sequence \( a_1 \subset \cdots \subset a_{r-1} \), and set \( s := h(r-1) \). By the second case, there exist elements \( t_{s+1}, \ldots, t_m \) satisfying (a) and (b) for \( k[u_{s+1}, \ldots, u_m] \) and \( a_r \cap k[u_{s+1}, \ldots, u_m] \). Then
\[
a_r \cap k[t_{s+1}, \ldots, t_m] = (t_{s+1}, \ldots, t_{h(r)})
\]
for some \( h(r) \). Set \( t_i := u_i \) for \( 1 \leq i \leq s \). Set \( R' := k[t_1, \ldots, t_m] \). Then \( R \) is module finite over \( k[u_1, \ldots, u_m] \) by hypothesis, and \( k[u_1, \ldots, u_m] \) is module finite over \( R' \) by hypothesis. Hence \( R \) is module finite over \( R' \). Thus (a) holds, and \( t_1, \ldots, t_m \) are algebraically independent over \( k \).

Fix \( i \) with \( 1 \leq i \leq r \). Set \( \ell := h(i) \). Then \( t_1, \ldots, t_\ell \in a_i \). Given \( x \in a_i \cap R' \), write \( x = \sum Q(v) t_1^{\nu_1} \cdots t_\ell^{\nu_\ell} \) with \( (v) = (v_1, \ldots, v_\ell) \in \mathbb{Z}_{\geq 0}^\ell \) and \( Q(v) \in k[t_{\ell+1}, \ldots, t_m] \). Then \( Q(0) \) lies in \( a_i \cap k[t_{\ell+1}, \ldots, t_m] \). The latter is equal to zero. It is zero if \( i \leq r - 1 \) because it lies in \( a_i \cap k[u_{\ell+1}, \ldots, u_m] \), which is equal to zero. and \( a_r \cap k[t_{\ell+1}, \ldots, t_m] \) is equal to \( (t_{\ell+1}, \ldots, t_m) \) by hypothesis. So \( a_r \cap k[t_{\ell+1}, \ldots, t_m] = 0 \). Thus \( Q(0) = 0 \). Hence \( x \in (t_1, \ldots, t_{h(i)}) \). Thus \( a_i \cap R' \) is contained in \( (t_1, \ldots, t_{h(i)}) \). So the two are equal. Thus (b) holds, and the theorem is proved.

**Remark (Another proof).** Suppose \( k \) is infinite. Then in the proof of the first case, we can take \( t_i := T_i - a_i T_1 \) for suitable \( a_i \in k \). Namely, say \( t_1 = H_d + \cdots + H_0 \) where \( H_i \) is homogeneous of degree \( i \) in \( T_1, \ldots, T_m \) and \( H_d \neq 0 \). Since \( k \) is infinite, there exist \( a_i \in k \) such that \( H_d(1, a_2, \ldots, a_m) \neq 0 \). Since \( H_d(1, a_2, \ldots, a_m) \) is the coefficient of \( T_1^d \) in
\[
H_d(T_1, t_2 + a_2 T_1, \ldots, t_m + a_m T_1),
\]
after collecting like powers of \( T_1 \), the equation
\[
t_1 - H_d(T_1, t_2 + a_2 T_1, \ldots, t_m + a_m T_1) - \cdots - H_0(T_1, t_2 + a_2 T_1, \ldots, t_m + a_m T_1) = 0
\]
becomes an equation of integral dependence of degree \( d \) for \( T_1 \) over \( k[t_1, \ldots, t_m] \).