2 General results of representation theory

2.1 Subrepresentations in semisimple representations

Let $A$ be an algebra.

Definition 2.1. A semisimple (or completely reducible) representation of $A$ is a direct sum of irreducible representations.

Example. Let $V$ be an irreducible representation of $A$ of dimension $n$. Then $Y = \text{End}(V)$, with action of $A$ by left multiplication, is a semisimple representation of $A$, isomorphic to $nV$ (the direct sum of $n$ copies of $V$). Indeed, any basis $v_1, ..., v_n$ of $V$ gives rise to an isomorphism of representations $\text{End}(V) \to nV$, given by $x \to (xv_1, ..., xv_n)$.

Remark. Note that by Schur’s lemma, any semisimple representation $V$ of $A$ is canonically identified with $\oplus_X \text{Hom}_A(X, V) \otimes X$, where $X$ runs over all irreducible representations of $A$. Indeed, we have a natural map $f : \oplus_X \text{Hom}(X, V) \otimes X \to V$, given by $g \otimes x \to g(x)$, $x \in X$, $g \in \text{Hom}(X, V)$, and it is easy to verify that this map is an isomorphism.

We’ll see now how Schur’s lemma allows us to classify subrepresentations in finite dimensional semisimple representations.

Proposition 2.2. Let $V_i, 1 \leq i \leq m$ be irreducible finite dimensional pairwise nonisomorphic representations of $A$, and $W$ be a subrepresentation of $V = \oplus_{i=1}^{m} n_i V_i$. Then $W$ is isomorphic to $\oplus_{i=1}^{m} r_i V_i$, $r_i \leq n_i$, and the inclusion $\phi : W \to V$ is a direct sum of inclusions $\phi_i : r_i V_i \to n_i V_i$ given by multiplication of a row vector of elements of $V_i$ (of length $r_i$) by a certain $r_i$-by-$n_i$ matrix $X_i$ with linearly independent rows: $\phi(v_1, ..., v_r) = (v_1, ..., v_r) X_i$.

Proof. The proof is by induction in $n := \sum_{i=1}^{m} n_i$. The base of induction ($n = 1$) is clear. To perform the induction step, let $W$ be nonzero, and fix an irreducible subrepresentation $P \subset W$. Such $P$ exists (Problem 1.20). ² Now, by Schur’s lemma, $P$ is isomorphic to $V_i$ for some $i$, and the inclusion $\phi|_P : P \to V$ factors through $n_i V_i$, and upon identification of $P$ with $V_i$ given by the formula $v \mapsto (v_1, ..., v_{q_i})$, where $q_i \in k$ are not all zero.

Now note that the group $G_i = GL_{n_i}(k)$ of invertible $n_i$-by-$n_i$ matrices over $k$ acts on $n_i V_i$ by $(v_1, ..., v_{n_i}) \mapsto (v_1, ..., v_{n_i}) g_i$ (and by the identity on $n_j V_j$, $j = i$), and therefore acts on the set of subrepresentations of $V_i$, preserving the property we need to establish: namely, under the action of $g_i$, the matrix $X_i$ goes to $X_i g_i$, while $X_j, j = i$ don’t change. Take $g_i \in G_i$ such that $(g_1, ..., g_{n_i}) g_i = (1, 0, ..., 0)$. Then $W g_i$ contains the first summand $V_i$ of $n_i V_i$ (namely, it is $P g_i$), hence $W g_i = V_i \oplus W'$, where $W' \subset n_1 V_1 \oplus \cdots \oplus (n_i - 1) V_i \oplus \cdots \oplus n_m V_m$ is the kernel of the projection of $W g_i$ to the first summand $V_i$ along the other summands. Thus the required statement follows from the induction assumption.

Remark 2.3. In Proposition 2.2, it is not important that $k$ is algebraically closed, nor it matters that $V$ is finite dimensional. If these assumptions are dropped, the only change needed is that the entries of the matrix $X_i$ are no longer in $k$ but in $D_i = \text{End}_A(V_i)$, which is, as we know, a division algebra. The proof of this generalized version of Proposition 2.2 is the same as before (check it!).

²Another proof of the existence of $P$, which does not use the finite dimensionality of $V$, is by induction in $n$. Namely, if $W$ itself is not irreducible, let $K$ be the kernel of the projection of $W$ to the first summand $V_1$. Then $K$ is a subrepresentation of $(n_1 - 1) V_1 \oplus \cdots \oplus n_m V_m$, which is nonzero since $W$ is not irreducible, so $K$ contains an irreducible subrepresentation by the induction assumption.
2.2 The density theorem

Let $A$ be an algebra over an algebraically closed field $k$.

**Corollary 2.4.** Let $V$ be an irreducible finite dimensional representation of $A$, and $v_1,\ldots,v_n \in V$ be any linearly independent vectors. Then for any $w_1,\ldots,w_n \in V$ there exists an element $a \in A$ such that $av_i = w_i$.

**Proof.** Assume the contrary. Then the image of the map $A \to nV$ given by $a \to (av_1,\ldots,av_n)$ is a proper subrepresentation, so by Proposition 2.2 it corresponds to an $r$-by-$n$ matrix $X$, $r < n$. Thus, taking $a = 1$, we see that there exist vectors $u_1,\ldots,u_r \in V$ such that $(u_1,\ldots,u_r)X = (v_1,\ldots,v_n)$. Let $(q_1,\ldots,q_n)$ be a nonzero vector such that $X(q_1,\ldots,q_n)^T = 0$ (it exists because $r < n$). Then $\sum q_i v_i = (u_1,\ldots,u_r)X(q_1,\ldots,q_n)^T = 0$, i.e. $\sum q_i v_i = 0$ - a contradiction with the linear independence of $v_i$. \hfill \Box

**Theorem 2.5.** (the Density Theorem). (i) Let $V$ be an irreducible finite dimensional representation of $A$. Then the map $\rho : A \to \text{End}V$ is surjective.

(ii) Let $V = V_1 \oplus \ldots \oplus V_r$, where $V_i$ are irreducible pairwise nonisomorphic finite dimensional representations of $A$. Then the map $\oplus_{i=1}^r \rho_i : A \to \oplus_{i=1}^r \text{End}(V_i)$ is surjective.

**Proof.** (i) Let $B$ be the image of $A$ in $\text{End}(V)$. We want to show that $B = \text{End}(V)$. Let $c \in \text{End}(V)$, $v_1,\ldots,v_n$ be a basis of $V$, and $w_i = cv_i$. By Corollary 2.4, there exists $a \in A$ such that $av_i = w_i$. Then $a$ maps to $c$, so $c \in B$, and we are done.

(ii) Let $B_i$ be the image of $A$ in $\text{End}(V_i)$, and $B$ be the image of $A$ in $\oplus_{i=1}^r \text{End}(V_i)$. Recall that as a representation of $A$, $\oplus_{i=1}^r \text{End}(V_i)$ is semisimple: it is isomorphic to $\oplus_{i=1}^r d_i V_i$, where $d_i = \dim V_i$. Then by Proposition 2.2, $B = \oplus_i B_i$. On the other hand, (i) implies that $B_i = \text{End}(V_i)$. Thus (ii) follows. \hfill \Box

2.3 Representations of direct sums of matrix algebras

In this section we consider representations of algebras $A = \bigoplus_i \text{Mat}_{d_i}(k)$ for any field $k$.

**Theorem 2.6.** Let $A = \bigoplus_{i=1}^r \text{Mat}_{d_i}(k)$. Then the irreducible representations of $A$ are $V_1 = k^{d_1},\ldots,V_r = k^{d_r}$, and any finite dimensional representation of $A$ is a direct sum of copies of $V_1,\ldots,V_r$.

In order to prove Theorem 2.6, we shall need the notion of a dual representation.

**Definition 2.7.** (Dual representation) Let $V$ be a representation of any algebra $A$. Then the dual representation $V^*$ is the representation of the opposite algebra $A^{\text{op}}$ (or, equivalently, right $A$-module) with the action

$$(f \cdot a)(v) := f(av).$$

**Proof of Theorem 2.6.** First, the given representations are clearly irreducible, as for any $v = 0, w \in V_i$, there exists $a \in A$ such that $av = w$. Next, let $X$ be an $n$-dimensional representation of $A$. Then, $X^*$ is an $n$-dimensional representation of $A^{\text{op}}$. But $(\text{Mat}_{d_i}(k))^{\text{op}} \cong \text{Mat}_{d_i}(k)$ with isomorphism $\varphi(X) = X^T$, as $(BC)^T = C^TB^T$. Thus, $A \cong A^{\text{op}}$ and $X^*$ may be viewed as an $n$-dimensional representation of $A$. Define

$$\phi : A \oplus \cdots \oplus A \to X^* \quad \text{with} \quad n \text{ copies}$$
by
\[ \phi(a_1, \ldots, a_n) = a_1y_1 + \cdots + a_ny_n \]
where \( \{y_i\} \) is a basis of \( X^* \). \( \phi \) is clearly surjective, as \( k \subset A \). Thus, the dual map \( \phi^* : X \to A^{n*} \) is injective. But \( A^{n*} \cong A^n \) as representations of \( A \) (check it!). Hence, \( \text{Im} \phi^* \cong X \) is a subrepresentation of \( A^n \). Next, \( \text{Mat}_d(k) = d_iV_i \), so \( A = \bigoplus_{i=1}^r d_iV_i \), \( A^n = \bigoplus_{i=1}^r nd_iV_i \), as a representation of \( A \). Hence by Proposition 2.2, \( X = \bigoplus_{i=1}^r m_iV_i \), as desired. \( \square \)

Exercise. The goal of this exercise is to give an alternative proof of Theorem 2.6, not using any of the previous results of Chapter 2.

Let \( A_1, A_2, \ldots, A_n \) be \( n \) algebras with units \( 1_1, 1_2, \ldots, 1_n \), respectively. Let \( A = A_1 \oplus A_2 \oplus \cdots \oplus A_n \). Clearly, \( 1\{1,1\}_j = \delta_{ij}1_i \), and the unit of \( A \) is \( 1 = 1_1 + 1_2 + \cdots + 1_n \).

For every representation \( V \) of \( A \), it is easy to see that \( 1_iV \) is a representation of \( A_i \) for every \( i \in \{1,2,\ldots,n\} \). Conversely, if \( V_1, V_2, \ldots, V_n \) are representations of \( A_1, A_2, \ldots, A_n \), respectively, then \( V_1 \oplus V_2 \oplus \cdots \oplus V_n \) canonically becomes a representation of \( A \) (with \( (a_1,a_2,\ldots,a_n) \in A \) acting on \( V_1 \oplus V_2 \oplus \cdots \oplus V_n \) as \( (v_1,v_2,\ldots,v_n) \mapsto (a_1v_1,a_2v_2,\ldots,a_nv_n) \)).

(a) Show that a representation \( V \) of \( A \) is irreducible if and only if \( 1_iV \) is an irreducible representation of \( A_i \) for exactly one \( i \in \{1,2,\ldots,n\} \), while \( 1_iV = 0 \) for all the other \( i \). Thus, classify the irreducible representations of \( A \) in terms of those of \( A_1, A_2, \ldots, A_n \).

(b) Let \( d \in \mathbb{N} \). Show that the only irreducible representation of \( \text{Mat}_d(k) \) is \( k^d \), and every finite dimensional representation of \( \text{Mat}_d(k) \) is a direct sum of copies of \( k^d \).

Hint: For every \( (i,j) \in \{1,2,\ldots,d\}^2 \), let \( E_{ij} \in \text{Mat}_d(k) \) be the matrix with 1 in the \( i \)th row of the \( j \)th column and 0’s everywhere else. Let \( V \) be a finite dimensional representation of \( \text{Mat}_d(k) \). Show that \( V = E_{11}V \oplus E_{22}V \oplus \cdots \oplus E_{dd}V \), and that \( \Phi_i : E_{11}V \to E_{ii}V, v \mapsto E_{11}v \) is an isomorphism for every \( i \in \{1,2,\ldots,d\} \). For every \( v \in E_{11}V \), denote \( S(v) = (E_{11}v,E_{21}v,\ldots,E_{d1}v) \). Prove that \( S(v) \) is a subrepresentation of \( V \) isomorphic to \( k^d \) (as a representation of \( \text{Mat}_d(k) \)), and that \( v \in S(v) \). Conclude that \( V = S(v_1) \oplus S(v_2) \oplus \cdots \oplus S(v_k) \), where \( \{v_1,v_2,\ldots,v_k\} \) is a basis of \( E_{11}V \).

(c) Conclude Theorem 2.6.

2.4 Filtrations

Let \( A \) be an algebra. Let \( V \) be a representation of \( A \). A (finite) filtration of \( V \) is a sequence of subrepresentations \( 0 = V_0 \subset V_1 \subset \cdots \subset V_n = V \).

Lemma 2.8. Any finite dimensional representation \( V \) of an algebra \( A \) admits a finite filtration \( 0 = V_0 \subset V_1 \subset \cdots \subset V_n = V \) such that the successive quotients \( V_i/V_{i-1} \) are irreducible.

Proof. The proof is by induction in \( \dim(V) \). The base is clear, and only the induction step needs to be justified. Pick an irreducible subrepresentation \( V_1 \subset V \), and consider the representation \( U = V/V_1 \). Then by the induction assumption \( U \) has a filtration \( 0 = U_0 \subset U_1 \subset \cdots \subset U_{n-1} = U \) such that \( U_i/U_{i-1} \) are irreducible. Define \( V_i \) for \( i \geq 2 \) to be the preimages of \( U_{i-1} \) under the tautological projection \( V \to V/V_1 = U \). Then \( 0 = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_n = V \) is a filtration of \( V \) with the desired property. \( \square \)
2.5 Finite dimensional algebras

**Definition 2.9.** The radical of a finite dimensional algebra $A$ is the set of all elements of $A$ which act by 0 in all irreducible representations of $A$. It is denoted $\text{Rad}(A)$.

**Proposition 2.10.** $\text{Rad}(A)$ is a two-sided ideal.

*Proof.* Easy. □

**Proposition 2.11.** Let $A$ be a finite dimensional algebra.

(i) Let $I$ be a nilpotent two-sided ideal in $A$, i.e., $I^n = 0$ for some $n$. Then $I \subset \text{Rad}(A)$.

(ii) $\text{Rad}(A)$ is a nilpotent ideal. Thus, $\text{Rad}(A)$ is the largest nilpotent two-sided ideal in $A$.

*Proof.* (i) Let $V$ be an irreducible representation of $A$. Let $v \in V$. Then $Iv \subset V$ is a subrepresentation. If $Iv = 0$ then $Iv = V$ so there is $x \in I$ such that $xv = v$. Then $x^n = 0$, a contradiction. Thus $Iv = 0$, so $I$ acts by 0 in $V$ and hence $I \subset \text{Rad}(A)$.

(ii) Let $0 = A_0 \subset A_1 \subset \ldots \subset A_n = A$ be a filtration of the regular representation of $A$ by subrepresentations such that $A_{i+1}/A_i$ are irreducible. It exists by Lemma 2.8. Let $x \in \text{Rad}(A)$. Then $x$ acts on $A_{i+1}/A_i$ by zero, so $x$ maps $A_{i+1}$ to $A_i$. This implies that $\text{Rad}(A)^n = 0$, as desired. □

**Theorem 2.12.** A finite dimensional algebra $A$ has only finitely many irreducible representations $V_i$ up to isomorphism, these representations are finite dimensional, and

$$A/\text{Rad}(A) \cong \bigoplus_i \text{End} V_i.$$ 

*Proof.* First, for any irreducible representation $V$ of $A$, and for any nonzero $v \in V$, $Av \subseteq V$ is a finite dimensional subrepresentation of $V$. (It is finite dimensional as $A$ is finite dimensional.) As $V$ is irreducible and $Av = 0$, $V = Av$ and $V$ is finite dimensional.

Next, suppose we have non-isomorphic irreducible representations $V_1, V_2, \ldots, V_r$. By Theorem 2.5, the homomorphism

$$\bigoplus_i \rho_i : A \rightarrow \bigoplus_i \text{End} V_i$$

is surjective. So $r \leq \sum_i \dim \text{End} V_i \leq \dim A$. Thus, $A$ has only finitely many non-isomorphic irreducible representations (at most $\dim A$).

Now, let $V_1, V_2, \ldots, V_r$ be all non-isomorphic irreducible finite dimensional representations of $A$. By Theorem 2.5, the homomorphism

$$\bigoplus_i \rho_i : A \rightarrow \bigoplus_i \text{End} V_i$$

is surjective. The kernel of this map, by definition, is exactly $\text{Rad}(A)$. □

**Corollary 2.13.** $\sum_i (\dim V_i)^2 \leq \dim A$, where the $V_i$’s are the irreducible representations of $A$.

*Proof.* As $\dim \text{End} V_i = (\dim V_i)^2$, Theorem 2.12 implies that $\dim A - \dim \text{Rad}(A) = \sum_i \dim \text{End} V_i = \sum_i (\dim V_i)^2$. As $\dim \text{Rad}(A) \geq 0$, $\sum_i (\dim V_i)^2 \leq \dim A$. □
Example 2.14. 1. Let \( A = k[x]/(x^n) \). This algebra has a unique irreducible representation, which is a 1-dimensional space \( k \), in which \( x \) acts by zero. So the radical \( \text{Rad}(A) \) is the ideal \( (x) \).

2. Let \( A \) be the algebra of upper triangular \( n \) by \( n \) matrices. It is easy to check that the irreducible representations of \( A \) are \( V_i, i = 1, \ldots, n \), which are 1-dimensional, and any matrix \( x \) acts by \( x_{ii} \). So the radical \( \text{Rad}(A) \) is the ideal of strictly upper triangular matrices (as it is a nilpotent ideal and contains the radical). A similar result holds for block-diagonal matrices.

Definition 2.15. A finite dimensional algebra \( A \) is said to be **semisimple** if \( \text{Rad}(A) = 0 \).

Proposition 2.16. For a finite dimensional algebra \( A \), the following are equivalent:

1. \( A \) is semisimple.

2. \( \sum_i (\dim V_i) = \dim A \), where the \( V_i \)'s are the irreducible representations of \( A \).

3. \( A \cong \bigoplus_i \text{Mat}_{d_i}(k) \) for some \( d_i \).

4. Any finite dimensional representation of \( A \) is completely reducible (that is, isomorphic to a direct sum of irreducible representations).

5. \( A \) is a completely reducible representation of \( A \).

Proof. As \( \dim A - \dim \text{Rad}(A) = \sum_i (\dim V_i) \), clearly \( \dim A = \sum_i (\dim V_i) \) if and only if \( \text{Rad}(A) = 0 \). Thus, (1) \( \iff \) (2).

Next, by Theorem 2.12, if \( \text{Rad}(A) = 0 \), then clearly \( A \cong \bigoplus_i \text{Mat}_{d_i}(k) \) for \( d_i = \dim V_i \). Thus, (1) \( \Rightarrow \) (3). Conversely, if \( A \cong \bigoplus_i \text{Mat}_{d_i}(k) \), then by Theorem 2.6, \( \text{Rad}(A) = 0 \), so \( A \) is semisimple. Thus (3) \( \Rightarrow \) (1).

Next, (3) \( \Rightarrow \) (4) by Theorem 2.6. Clearly (4) \( \Rightarrow \) (5). To see that (5) \( \Rightarrow \) (3), let \( A = \bigoplus_i n_i V_i \). Consider \( \text{End}_A(A) \) (endomorphisms of \( A \) as a representation of \( A \)). As the \( V_i \)'s are pairwise non-isomorphic, by Schur's lemma, no copy of \( V_i \) in \( A \) can be mapped to a distinct \( V_j \). Also, again by Schur's lemma, \( \text{End}_A(V_i) = k \). Thus, \( \text{End}_A(A) \cong \bigoplus_i \text{Mat}_{n_i}(k) \). But \( \text{End}_A(A) \cong A^{\text{op}} \) by Problem 1.22, so \( A^{\text{op}} \cong \bigoplus_i \text{Mat}_{n_i}(k) \). Thus, \( A \cong (\bigoplus_i \text{Mat}_{n_i}(k))^{\text{op}} = \bigoplus_i \text{Mat}_{n_i}(k) \), as desired. \( \square \)

2.6 Characters of representations

Let \( A \) be an algebra and \( V \) a finite-dimensional representation of \( A \) with action \( \rho \). Then the **character** of \( V \) is the linear function \( \chi_V : A \to k \) given by

\[
\chi_V(a) = \text{tr}|_V(\rho(a)).
\]

If \( [A, A] \) is the span of commutators \( [x, y] := xy - yx \) over all \( x, y \in A \), then \( [A, A] \subseteq \ker \chi_V \). Thus, we may view the character as a mapping \( \chi_V : A/[A, A] \to k \).

**Exercise.** Show that if \( W \subset V \) are finite dimensional representations of \( A \), then \( \chi_V = \chi_W + \chi_{V/W} \).

Theorem 2.17. (i) Characters of (distinct) irreducible finite-dimensional representations of \( A \) are linearly independent.

(ii) If \( A \) is a finite-dimensional semisimple algebra, then these characters form a basis of \( (A/[A, A])^* \).
Proof. (i) If $V_1, \ldots, V_r$ are nonisomorphic irreducible finite-dimensional representations of $A$, then $\rho V_1 \oplus \cdots \oplus \rho V_r : A \to \text{End} V_1 \oplus \cdots \oplus \text{End} V_r$ is surjective by the density theorem, so $\chi_{V_1}, \ldots, \chi_{V_r}$ are linearly independent. (Indeed, if $\sum \lambda_i \chi_{V_i}(a) = 0$ for all $a \in A$, then $\sum \lambda_i \text{Tr}(M_i) = 0$ for all $M_i \in \text{End} V_i$. But each $\text{Tr}(M_i)$ can range independently over $k$, so it must be that $\lambda_1 = \cdots = \lambda_r = 0$.)

(ii) First we prove that $[\text{Mat}_d(k), \text{Mat}_d(k)] = \text{sl}_d(k)$, the set of all matrices with trace 0. It is clear that $[\text{Mat}_d(k), \text{Mat}_d(k)] \subseteq \text{sl}_d(k)$. If we denote by $E_{ij}$ the matrix with 1 in the $i$th row of the $j$th column and 0’s everywhere else, we have $[E_{ij}, E_{jm}] = E_{im}$ for $i = m$, and $[E_{i,i+1}, E_{i+1,i}] = E_{ii} - E_{i+1,i+1}$. Now $\{E_{im}\} \cup \{E_{ii} - E_{i+1,i+1}\}$ forms a basis in $\text{sl}_d(k)$, so indeed $[\text{Mat}_d(k), \text{Mat}_d(k)] = \text{sl}_d(k)$, as claimed.

By semisimplicity, we can write $A = \text{Mat}_{d_1}(k) \oplus \cdots \oplus \text{Mat}_{d_r}(k)$. Then $[A, A] = \text{sl}_{d_1}(k) \oplus \cdots \oplus \text{sl}_{d_r}(k)$, and $A/[A, A] \cong k^r$. By Theorem 2.6, there are exactly $r$ irreducible representations of $A$ (isomorphic to $k^{d_1}, \ldots, k^{d_r}$, respectively), and therefore $r$ linearly independent characters on the $r$-dimensional vector space $A/[A, A]$. Thus, the characters form a basis.

2.7 The Jordan-Hölder theorem

We will now state and prove two important theorems about representations of finite dimensional algebras - the Jordan-Hölder theorem and the Krull-Schmidt theorem.

Theorem 2.18. (Jordan-Hölder theorem). Let $V$ be a finite dimensional representation of $A$, and $0 = V_0 \subset V_1 \subset \ldots \subset V_n = V$, $0 = V'_0 \subset \ldots \subset V'_m = V$ be filtrations of $V$, such that the representations $W_i := V_i/V_{i-1}$ and $W'_i := V'_i/V'_{i-1}$ are irreducible for all $i$. Then $n = m$, and there exists a permutation $\sigma$ of $1, \ldots, n$ such that $W_{\sigma(i)}$ is isomorphic to $W'_i$.

Proof. First proof (for $k$ of characteristic zero). The character of $V$ obviously equals the sum of characters of $W_i$, and also the sum of characters of $W'_i$. But by Theorem 2.17, the characters of irreducible representations are linearly independent, so the multiplicity of every irreducible representation $W$ of $A$ among $W_i$ and among $W'_i$ are the same. This implies the theorem. 3

Second proof (general). The proof is by induction on $\dim V$. The base of induction is clear, so let us prove the induction step. If $W_1 = W'_l$ (as subspaces), we are done, since by the induction assumption the theorem holds for $V/W_1$. So assume $W_1 \neq W'_1$. In this case $W_1 \cap W'_1 = 0$ (as $W_1, W'_1$ are irreducible), so we have an embedding $f : W_1 \oplus W'_1 \to V$. Let $U = V/(W_1 \oplus W'_1)$, and $0 = U_0 \subset U_1 \subset \ldots \subset U_p = U$ be a filtration of $U$ with simple quotients $Z_i = U_i/U_{i-1}$ (it exists by Lemma 2.8). Then we see that:

1) $V/W_1$ has a filtration with successive quotients $W'_1, Z_1, \ldots, Z_p$, and another filtration with successive quotients $W_2, \ldots, W_n$.

2) $V/W'_1$ has a filtration with successive quotients $W_1, Z_1, \ldots, Z_p$, and another filtration with successive quotients $W'_2, \ldots, W'_n$.

By the induction assumption, this means that the collection of irreducible representations with multiplicities $W_1, W'_1, Z_1, \ldots, Z_p$ coincides on one hand with $W_1, \ldots, W_n$, and on the other hand, with $W'_1, \ldots, W'_m$. We are done.

The Jordan-Hölder theorem shows that the number $n$ of terms in a filtration of $V$ with irreducible successive quotients does not depend on the choice of a filtration, and depends only on

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3This proof does not work in characteristic $p$ because it only implies that the multiplicities of $W_i$ and $W'_i$ are the same modulo $p$, which is not sufficient. In fact, the character of the representation $pV$, where $V$ is any representation, is zero.
V. This number is called the length of V. It is easy to see that n is also the maximal length of a filtration of V in which all the inclusions are strict.

The sequence of the irreducible representations \( W_1, \ldots, W_n \) enumerated in the order they appear from some filtration of V as successive quotients is called a Jordan-Hölder series of V.

### 2.8 The Krull-Schmidt theorem

**Theorem 2.19.** (Krull-Schmidt theorem) Any finite dimensional representation of A can be uniquely (up to an isomorphism and order of summands) decomposed into a direct sum of indecomposable representations.

**Proof.** It is clear that a decomposition of V into a direct sum of indecomposable representations exists, so we just need to prove uniqueness. We will prove it by induction on \( \dim V \). Let \( V = V_1 \oplus \ldots \oplus V_m = V'_1 \oplus \ldots \oplus V'_n \). Let \( i_s : V_s \to V, i'_s : V'_s \to V, p_s : V \to V_s, p'_s : V \to V'_s \) be the natural maps associated to these decompositions. Let \( \theta_s = p_1 i'_s p'_s i_1 : V_1 \to V_1 \). We have \( \sum_{s=1}^n \theta_s = 1 \). Now we need the following lemma.

**Lemma 2.20.** Let \( W \) be a finite dimensional indecomposable representation of A. Then

(i) Any homomorphism \( \theta : W \to W \) is either an isomorphism or nilpotent;

(ii) If \( \theta_s : W \to W, s = 1, \ldots, n \) are nilpotent homomorphisms, then so is \( \theta := \theta_1 + \ldots + \theta_n \).

**Proof.** (i) Generalized eigenspaces of \( \theta \) are subrepresentations of \( W \), and \( W \) is their direct sum. Thus, \( \theta \) can have only one eigenvalue \( \lambda \). If \( \lambda \) is zero, \( \theta \) is nilpotent, otherwise it is an isomorphism.

(ii) The proof is by induction in \( n \). The base is clear. To make the induction step (\( n - 1 \) to \( n \)), assume that \( \theta \) is not nilpotent. Then by (i) \( \theta \) is an isomorphism, so \( \sum_{i=1}^n \theta^{-1} \theta_i = 1 \). The morphisms \( \theta^{-1} \theta_i \) are not isomorphisms, so they are nilpotent. Thus \( 1 - \theta^{-1} \theta_n = \theta^{-1} \theta_1 + \ldots + \theta^{-1} \theta_{n-1} \) is an isomorphism, which is a contradiction with the induction assumption.

By the lemma, we find that for some \( s, \theta_s \) must be an isomorphism; we may assume that \( s = 1 \). In this case, \( V'_1 = \text{Im}(p'_1 i_1) \oplus \text{Ker}(p_1 i'_1) \), so since \( V'_1 \) is indecomposable, we get that \( f := p'_1 i_1 : V_1 \to V'_1 \) and \( g := p_1 i'_1 : V'_1 \to V_1 \) are isomorphisms.

Let \( B = \bigoplus_{j>1} V_j, B' = \bigoplus_{j>1} V'_j \); then we have \( V = V_1 \oplus B = V'_1 \oplus B' \). Consider the map \( h : B \to B' \) defined as a composition of the natural maps \( B \to V \to B' \) attached to these decompositions. We claim that \( h \) is an isomorphism. To show this, it suffices to show that \( \text{Ker} h = 0 \) (as \( h \) is a map between spaces of the same dimension). Assume that \( v \in \text{Ker} h \subset B \). Then \( v \in V'_1 \). On the other hand, the projection of \( v \) to \( V_1 \) is zero, so \( g v = 0 \). Since \( g \) is an isomorphism, we get \( v = 0 \), as desired.

Now by the induction assumption, \( m = n \), and \( V_j \cong V'_{\sigma(j)} \) for some permutation \( \sigma \) of \( 2, \ldots, n \). The theorem is proved.

**Exercise.** Let \( A \) be the algebra of real-valued continuous functions on \( \mathbb{R} \) which are periodic with period 1. Let \( M \) be the \( A \)-module of continuous functions \( f \) on \( \mathbb{R} \) which are antiperiodic with period 1, i.e., \( f(x + 1) = -f(x) \).

(i) Show that \( A \) and \( M \) are indecomposable \( A \)-modules.

(ii) Show that \( A \) is not isomorphic to \( M \) but \( A \oplus A \) is isomorphic to \( M \oplus M \).
Remark. Thus, we see that in general, the Krull-Schmidt theorem fails for infinite dimensional modules. However, it still holds for modules of finite length, i.e., modules $M$ such that any filtration of $M$ has length bounded above by a certain constant $l = l(M)$.

2.9 Problems

Problem 2.21. Extensions of representations. Let $A$ be an algebra, and $V,W$ be a pair of representations of $A$. We would like to classify representations $U$ of $A$ such that $V$ is a subrepresentation of $U$, and $U/V = W$. Of course, there is an obvious example $U = V ⊕ W$, but are there any others?

Suppose we have a representation $U$ as above. As a vector space, it can be (non-uniquely) identified with $V ⊕ W$, so that for any $a ∈ A$ the corresponding operator $\rho_U(a)$ has block triangular form

$$\rho_U(a) = \begin{pmatrix} \rho_V(a) & f(a) \\ 0 & \rho_W(a) \end{pmatrix},$$

where $f : A → \text{Hom}_k(W,V)$ is a linear map.

(a) What is the necessary and sufficient condition on $f(a)$ under which $\rho_U(a)$ is a representation? Maps $f$ satisfying this condition are called $(1,1)$-cocycles (of $A$ with coefficients in $\text{Hom}_k(W,V)$). They form a vector space denoted $Z^1(W,V)$.

(b) Let $X : W → V$ be a linear map. The coboundary of $X$, $dX$, is defined to be the function $A → \text{Hom}_k(W,V)$ given by $dX(a) = \rho_V(a)X - XP_X(a)$. Show that $dX$ is a cocycle, which vanishes if and only if $X$ is a homomorphism of representations. Thus coboundaries form a subspace $B^1(W,V) ⊂ Z^1(W,V)$, which is isomorphic to $\text{Hom}_k(W,V)/\text{Hom}_A(W,V)$. The quotient $Z^1(W,V)/B^1(W,V)$ is denoted $\text{Ext}^1(W,V)$.

(c) Show that if $f, f' ∈ Z^1(W,V)$ and $f - f' ∈ B^1(W,V)$ then the corresponding extensions $U, U'$ are isomorphic representations of $A$. Conversely, if $φ : U → U'$ is an isomorphism such that

$$φ(a) = \begin{pmatrix} 1_V & * \\ 0 & 1_W \end{pmatrix}$$

then $f - f' ∈ B^1(W,V)$. Thus, the space $\text{Ext}^1(W,V)$ “classifies” extensions of $W$ by $V$.

(d) Assume that $W,V$ are finite dimensional irreducible representations of $A$. For any $f ∈ \text{Ext}^1(W,V)$, let $U_f$ be the corresponding extension. Show that $U_f$ is isomorphic to $U_{f'}$ as representations if and only if $f$ and $f'$ are proportional. Thus isomorphism classes (as representations) of nontrivial extensions of $W$ by $V$ (i.e., those not isomorphic to $W ⊕ V$) are parametrized by the projective space $\mathbb{P}\text{Ext}^1(W,V)$. In particular, every extension is trivial if and only if $\text{Ext}^1(W,V) = 0$.

Problem 2.22. (a) Let $A = C[x_1, ..., x_n]$, and $V_a, V_b$ be one-dimensional representations in which $x_i$ act by $a_i$ and $b_i$, respectively $(a_i, b_i ∈ C)$. Find $\text{Ext}^1(V_a, V_b)$ and classify 2-dimensional representations of $A$.

(b) Let $B$ be the algebra over $C$ generated by $x_1, ..., x_n$ with the defining relations $x_i x_j = 0$ for all $i, j$. Show that for $n > 1$ the algebra $B$ has infinitely many non-isomorphic indecomposable representations.

Problem 2.23. Let $Q$ be a quiver without oriented cycles, and $P_Q$ the path algebra of $Q$. Find irreducible representations of $P_Q$ and compute $\text{Ext}^1$ between them. Classify 2-dimensional representations of $P_Q$. 
Problem 2.24. Let $A$ be an algebra, and $V$ a representation of $A$. Let $\rho : A \to \text{End}V$. A formal deformation of $V$ is a formal series

$$\tilde{\rho} = \rho_0 + t\rho_1 + ... + t^n\rho_n + ...,$$

where $\rho_i : A \to \text{End}(V)$ are linear maps, $\rho_0 = \rho$, and $\tilde{\rho}(ab) = \tilde{\rho}(a)\tilde{\rho}(b)$.

If $b(t) = 1 + b_1 t + b_2 t^2 + ...$, where $b_i \in \text{End}(V)$, and $\tilde{\rho}$ is a formal deformation of $\rho$, then $b\tilde{\rho}b^{-1}$ is also a deformation of $\rho$, which is said to be isomorphic to $\tilde{\rho}$.

(a) Show that if $\text{Ext}^1(V,V) = 0$, then any deformation of $\rho$ is trivial, i.e., isomorphic to $\rho$.

(b) Is the converse to (a) true? (consider the algebra of dual numbers $A = k[x]/x^2$).

Problem 2.25. The Clifford algebra. Let $V$ be a finite dimensional complex vector space equipped with a symmetric bilinear form $(,)$. The Clifford algebra $\text{Cl}(V)$ is the quotient of the tensor algebra $TV$ by the ideal generated by the elements $v \otimes v - (v,v)1$, $v \in V$. More explicitly, if $x_i, 1 \leq i \leq N$ is a basis of $V$ and $(x_i,x_j) = a_{ij}$ then $\text{Cl}(V)$ is generated by $x_i$ with defining relations

$$x_i x_j + x_j x_i = 2a_{ij}, x_i^2 = a_{ii}.$$  

Thus, if $(,)$ is 0, $\text{Cl}(V) = \wedge V$.

(i) Show that if $(,)$ is nondegenerate then $\text{Cl}(V)$ is semisimple, and has one irreducible representation of dimension $2^n$ if $\dim V = 2n$ (so in this case $\text{Cl}(V)$ is a matrix algebra), and two such representations if $\dim(V) = 2n + 1$ (i.e., in this case $\text{Cl}(V)$ is a direct sum of two matrix algebras).

Hint. In the even case, pick a basis $a_1, ..., a_n, b_1, ..., b_n$ of $V$ in which $(a_i,a_j) = (b_i,b_j) = 0$, $(a_i,b_j) = \delta_{ij}/2$, and construct a representation of $\text{Cl}(V)$ on $S := \wedge (a_1, ..., a_n)$ in which $b_i$ acts as “differentiation” with respect to $a_i$. Show that $S$ is irreducible. In the odd case the situation is similar, except there should be an additional basis vector $c$ such that $(c,a_i) = (c,b_i) = 0$, $(c,c) = 1$, and the action of $c$ on $S$ may be defined either by $(-1)^{\text{degree}}$ or by $(-1)^{\text{degree} + 1}$, giving two representations $S_+, S_-$ (why are they non-isomorphic?). Show that there is no other irreducible representations by finding a spanning set of $\text{Cl}(V)$ with $2^{\dim V}$ elements.

(ii) Show that $\text{Cl}(V)$ is semisimple if and only if $(,)$ is nondegenerate. If $(,)$ is degenerate, what is $\text{Cl}(V)/\text{Rad}(\text{Cl}(V))$?

2.10 Representations of tensor products

Let $A, B$ be algebras. Then $A \otimes B$ is also an algebra, with multiplication $(a_1 \otimes b_1)(a_2 \otimes b_2) = a_1a_2 \otimes b_1b_2$.

Exercise. Show that $\text{Mat}_m(k) \otimes \text{Mat}_n(k) \cong \text{Mat}_{mn}(k)$.

The following theorem describes irreducible finite dimensional representations of $A \otimes B$ in terms of irreducible finite dimensional representations of $A$ and those of $B$.

Theorem 2.26. (i) Let $V$ be an irreducible finite dimensional representation of $A$ and $W$ an irreducible finite dimensional representation of $B$. Then $V \otimes W$ is an irreducible representation of $A \otimes B$.

(ii) Any irreducible finite dimensional representation $M$ of $A \otimes B$ has the form (i) for unique $V$ and $W$.

Remark 2.27. Part (ii) of the theorem typically fails for infinite dimensional representations; e.g. it fails when $A$ is the Weyl algebra in characteristic zero. Part (i) also may fail. E.g. let $A = B = V = W = \mathbb{C}(x)$. Then (i) fails, as $A \otimes B$ is not a field.
Proof. (i) By the density theorem, the maps $A \to \text{End} V$ and $B \to \text{End} W$ are surjective. Therefore, the map $A \otimes B \to \text{End} V \otimes \text{End} W = \text{End}(V \otimes W)$ is surjective. Thus, $V \otimes W$ is irreducible.

(ii) First we show the existence of $V$ and $W$. Let $A', B'$ be the images of $A, B$ in $\text{End} M$. Then $A', B'$ are finite dimensional algebras, and $M$ is a representation of $A' \otimes B'$, so we may assume without loss of generality that $A$ and $B$ are finite dimensional.

In this case, we claim that $\text{Rad}(A \otimes B) = \text{Rad}(A) \otimes B + A \otimes \text{Rad}(B)$. Indeed, denote the latter by $J$. Then $J$ is a nilpotent ideal in $A \otimes B$, as $\text{Rad}(A)$ and $\text{Rad}(B)$ are nilpotent. On the other hand, $(A \otimes B)/J = (A/\text{Rad}(A)) \otimes (B/\text{Rad}(B))$, which is a product of two semisimple algebras, hence semisimple. This implies $J \supset \text{Rad}(A \otimes B)$. Altogether, by Proposition 2.11, we see that $J = \text{Rad}(A \otimes B)$, proving the claim.

Thus, we see that

$$(A \otimes B)/\text{Rad}(A \otimes B) = A/\text{Rad}(A) \otimes B/\text{Rad}(B).$$

Now, $M$ is an irreducible representation of $(A \otimes B)/\text{Rad}(A \otimes B)$, so it is clearly of the form $M = V \otimes W$, where $V$ is an irreducible representation of $A/\text{Rad}(A)$ and $W$ is an irreducible representation of $B/\text{Rad}(B)$, and $V, W$ are uniquely determined by $M$ (as all of the algebras involved are direct sums of matrix algebras).
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