Recall. If $\Sigma_1 \subseteq k^{n_1}$ and $\Sigma_2 \subseteq k^{n_2}$ are algebraic subsets, then a morphism $\rho : \Sigma_1 \to \Sigma_2$ is a set map such that there exist polynomials $f_1, \ldots, f_{n_2} \in k[x_1, \ldots, x_{n_2}]$ so that $\rho(\underline{a}) = (f_1(\underline{a}), \ldots, f_{n_2}(\underline{a}))$.

**Thm. 9** There is a natural bijection $\text{Hom}(\Sigma_1, \Sigma_2) \cong \text{Hom}_{k-Alg}(R_2, R_1)$ where $R_i = k[x_1, \ldots, x_{n_i}] / I(\Sigma_i)$.

**Proof.** Define $\alpha : \text{Hom}(\Sigma_1, \Sigma_2) \to \text{Hom}(R_2, R_1)$. Given $\rho$, choose $f_1, \ldots, f_{n_2}$ inducing $\rho$ and use the map $\tau$ between $k[x_1, \ldots, x_{n_2}]$ to $k[x_1, \ldots, x_{n_1}]$ defined by $x_i \mapsto f_i$.

**Claim.** This extends naturally to a map between $R_2$ and $R_1$. To construct this, we wish to find $\rho^*$ such that $\rho^*$ commutes with $\tau$ and the surjective maps onto $R_2$ and $R_1$. In order to prove that $\rho^*$ exists, we need to verify that if $g(x_1, \ldots, x_{n_2}) \in I(\Sigma_2)$ then $g(f_1(\underline{x}), \ldots, f_{n_2}(\underline{x})) \in I(\Sigma_1)$. Say that $\underline{a} \in \Sigma_1$. Then this is $g(\rho(\underline{a})) = 0$.

We also wish to show that $\rho^*$ is unique, independent of the choices that induce $\tau$. Thus, suppose $\tau'$ is another such map, defined by a second set of choices, $f_1', \ldots, f_{n_2}'$. Then we need to prove that $\tau(x_i) - \tau'(x_i) \in I(\Sigma_1)$. (This will show that the map is natural.) All we need to do is note that $f_i' = f_i + g_i$ for some $g_i \in I(\Sigma_1)$, and we are done. (?)

This all shows that $\alpha$ is well defined; it maps $\rho$ to $\rho^*$. To show that $\alpha$ is injective, we must show how to construct $\rho$ from $\rho^* = \alpha(\rho)$. $\rho : \Sigma_1 \to \Sigma_2$ is given by $m \subseteq R_1 \mapsto \rho^{-1}(m) \subseteq R_2$ (this is a map on maximal ideals). This works.

To show that $\alpha$ is surjective, consider a given $\rho^*$ in $\text{Hom}(R_2, R_1)$ and we will try to lift this to a $\rho \in \text{Hom}(\Sigma_1, \Sigma_2)$. So to do this, we use a commutative diagram, by lifting $\rho^*$ to $\tau$ and then to a map from $\Sigma_1$ to $\Sigma_2$ defined by $(a_1, \ldots, a_{n_1}) \mapsto (\tau(\underline{x})(\underline{a}), \ldots, \tau(x_{n_2})(\underline{a}))$.

Martin says, study this until it's very clear. This is really a diagram argument.

**Example:**

$\Sigma_1$ is the parabola $y = x^2$ and $\Sigma_2$ is the line $y = 0$. Then, the only natural morphism takes $x$ to $x$, while taking $0$ to $x^2$. This produces a map from $k[x, y]/(y)$ to $k[x, y]/(y-x^2)$.

**Example:** $k = \mathbb{A} \to C = \{(a, a^2, a^3) | a \in k\} \subseteq k^3$. Then, we get a map

$k[X, Y, Z]/(Y - X^2, Z - X^3) \to k[T]$ defined by $X \mapsto T, Y \mapsto T^2, Z \mapsto T^3$. The inverse map is $T \mapsto X$.

**Def.** If $\Sigma \subseteq k^n$ is an alg. set, then the affine coordinate ring of $\Sigma$ is $\Gamma(\Sigma) = k[x_1, \ldots, x_n] / I(\Sigma)$. This will, in a sense, be independent of the embedding of $\Sigma$ in $k^n$.

**Lemma** There is a natural bijection $\Gamma \Sigma \cong \text{Hom}(\Sigma, \mathbb{A})$.

**Pf.** $\text{Hom}(\Sigma, \mathbb{A}) \cong \text{Hom}(k[T], \Gamma(\Sigma)) \cong \Gamma(\Sigma)$, where this last is defined by $\rho^* \mapsto \rho^*(T)$. The first part is given by our previous theorem.

**Goal:** $\Gamma$ induces an equivalence of categories between the category of irreducible affine algebraic sets and the category of integral $k$-algebras of finite type.

Now, a brief divergence into category theory.

**Def.** A category $\mathcal{C}$ is a set of “objects” $\text{Ob}(\mathcal{C})$. For any two objects $X, Y \in \text{Ob}(\mathcal{C})$ there is a set $\text{Hom}_\mathcal{C}(X, Y)$ called the set of morphisms from $X$ to $Y$. Further, for $X, Y, Z \in \text{Ob}(\mathcal{C})$, a composition map $\text{Hom}(X, Y) \times \text{Hom}(Y, Z) \to \text{Hom}(X, Z)$ is defined such that
1. \( \forall X \in Ob(\mathcal{C}) \) there is an \( id_X \in \text{Hom}(X, X) \) which is a right (left) identity for elements in \( \text{Hom}(Y, X) \) (\( \text{Hom}(X, Y) \)).

2. Composition is associative.

**Example.** The categories of affine algebraic sets, modules over a given ring, and \( k \)-algebras are sensible categories. The maps are defined just as they should be.

**Def.** If \( \mathcal{C} \) and \( \mathcal{D} \) are categories, a contravariant functor \( F : \mathcal{C} \to \mathcal{D} \) is an association where for each \( A \in Ob(\mathcal{C}) \) we get a particular \( F(A) \in Ob(\mathcal{D}) \). Also, for all \( f \in \text{Hom}_\mathcal{C}(X, Y) \) we get a map \( F(f) : F(Y) \to F(X) \), preserving composition and identity.

A covariant functor gives, from \( f \), a map \( F(f) : F(X) \to F(Y) \).

To return to our goal, the functor that takes \( \Sigma \) to \( \Gamma(\Sigma) \) and that takes morphisms to \( k \)-algebra maps is a covariant functor. We have already proved all the parts we need for this.

**Def.** \( F \) is an equivalence if there is an “inverse” functor \( F^{-1} \) such that for every object \( A \in \mathcal{D} \) is isomorphic to \( F(X) \) for some \( X \in \mathcal{C} \) and this map is \( F^{-1} \). Furthermore, for all \( X, Y \in \mathcal{C} \) the map \( \text{Hom}_\mathcal{C}(X, Y) \xrightarrow{F} \text{Hom}_\mathcal{D}(F(Y), F(X)) \) is also bijective.

To show that \( \Gamma \) is an equivalence, we need to note that both the maps of morphisms and the maps of objects are bijective. We have already shown this for the morphisms. Suppose \( R \) is an integral \( k \)-algebra of finite type, meaning \( R = k[x_1, \ldots, x_n]/P \) for some prime ideal \( P \).

Remark: this equivalence fails for algebraic sets in \( \mathbb{P}^n \).