1 presheaves

Recall. A presheaf on a topological space \(X\) is a contravariant functor \(F : \text{Op}(X) \to \text{Set}\) where \(V \mapsto F(V)\).

Example: \(X, Y\) contravariant spaces, then if \(F(V) = \{\text{cont. maps } U \to Y\}\) defines a presheaf.

Def. A presheaf \(F\) is a sheaf if for all collections \(\{U_i\}\) of open sets the following sequence is exact:

\[
F(\bigcup_i U_i) \xrightarrow{j} \Pi_i F(U_i) \to \Pi_{i,j} F(U_i \cap U_j).
\]

Recall that a diagram of sets \(S \xrightarrow{j} T \xrightarrow{p} R\) if \(j\) is injective and if for any \(t \in T\) with \(p_1(t) = p_2(t)\) there exists an \(s \in S\) with \(j(s) = t\).

This is rather confusing. What we mean is that the maps glue together.

That is, \(f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}\).

If \(S, T, R\) were also group then \(S \xrightarrow{j} T \xrightarrow{p} R\) exact means \(0 \to S \xrightarrow{j} T \xrightarrow{p} R\) is exact. A restatement.

1. If \(x_1, x_2 \in F(U)\) and \(\text{res}_{U_i} x_1 = \text{res}_{U_i} x_2\) for all \(i\), then \(x_1 = x_2\), and

2. Given \(x_i \in F(U_i)\) s.t. \(\text{res}_{U_i \cap U_j} (x_i) = \text{res}_{U_i \cap U_j} (x_j) \forall i, j\) then there is an \(x \in F(U)\) such that \(x_i = \text{res}_{U_i} (x) \forall i\).

Example. Let \(X, Y\) be topological spaces. Then \(F(U) = \{\text{continuous maps } U \to Y\}\) is a sheaf.

(1) is clear.

(2) We construct the map from \(X\) to \(Y\) by \(x \mapsto f_i(x)\) for any \(i\) for which \(x \in U_i\). This is unambiguous since these maps agree on the intersections, and we know this is continuous.

Ex. Let \(f : X \to Y\) be continuous. Let \(\mathcal{F}\) be a sheaf on \(X\) and define \((f_* \mathcal{F})(V) = \mathcal{F}(f^{-1}(V))\). Or, to put it another way, we have

\[\text{Op}(Y) \xrightarrow{f^{-1}} \text{Op}(X) \xrightarrow{\mathcal{F}} \text{Set},\]

so the composition \(\text{Op}(Y) \xrightarrow{f_*} \text{Set}\) is the sheaf we’re looking for. Let \(V = \bigcup V_i, U = f^{-1}(V), U_i = f^{-1}V_i\).

Now, we want exactness for

\[f_* \mathcal{F}(V) \to \Pi_i f_* \mathcal{F}(V_i) \to \Pi_{i,j} f_* \mathcal{F}(V_i \cap V_j),\]

but these are just \(\mathcal{F}(V) \to \pi_* \mathcal{F}(U_i) \to \pi_{i,j} \mathcal{F}(U_i \cap U_j)\), which is exact since \(\mathcal{F}\) is a sheaf.

Ex. (presheaf but not a sheaf.) Suppose we have \(X, S \neq \{\ast\}\), a set, such that \(F(U) = S\) for every \(U\). We claim that \(F(\emptyset)\) is a one-element set (axiom?!) so this would only be a sheaf if \(S \cong S \times S\), obviously not the case if \(S\) is finite but larger than 1 element.
Ex. Let $X$ be a top. mspace, $G$ a finite group with the discrete topology, and let $F(U) = \{\text{cont. maps } U \to G\}/\{\text{constant maps } U \to G\}$. The point is that if $X$ is not connected, then you might have a constant map on each connected component that is not a constant map globally, so $F(U) \to \bigoplus_i F(U_i)$ is not injective.

Theorem. Let $X$ be a top. space, $F$ a presheaf. Then $\exists$ a sheaf $F^a$ with a map $F \to F^a$ which is universal for maps to sheaves. That is, if we have a map $F \to G$ of presheaves, there is a unique map $F^a \to G$ of sheaves. (From this, it automatically follows that $F^a$ is unique up to isomorphism.

Def. If $F$ is a presheaf, $x \in X$, then the stalk $F_x$ at $x$ is defined to be $\varinjlim_{U \ni x} F(U)$. This is the disjoint union over all $x \in U$ modulo that two elements are equivalent if they agree on restrictions to smaller and smaller neighborhoods of $x$.

Define $F^a(U) = \{(f_x)_{x \in U}, f_x \in F_x\}$, such that there is a $U_i$ such that $\cup U_i = U$, and $f_i \in F(U_i)$ inducing $(f_x)_{x \in U_i}$. It is clear $F^a$ is a sheaf: it was made to be a sheaf.

Ex. $X = \{a, b\}$ with the discrete topology, and the presheaf $F(U) = S$ for all $U$. Let $U_1 = \{a\}$ and $U_2 = \{b\}$. $F_a = F(U_1) = S$. Also, $F_b = F(U_2) = S$. $F^a(U_1) = S$. Similarly, $F^a(U_2) = S$. What is $F^a(U_1 \cup U_2)$? It must be $S \times S$ by our construction. This is no longer the constant presheaf, but is actually a sheaf.

Now, let us say we have a map $\alpha : F \to G$ of presheaves. We will now construct a map $F^a(U) \to G(U)$ that is a map of sheaves. We know how to map $(f_x)_{x \in U}$ into $\prod_i G(U_i)$: we map it to $\alpha(f_i)$. We need to check that $\alpha(f_i) = \alpha(f_j)$ on $G(U_i \cap U_j)$.

Ex. $X$ a locally connected top. space, $F(U) = S$. Then $F^a(U)$ is the disjoint union over connected components of $U$ of $S$.

Notation Let $F$ be a sheaf on $X$. We write $\Gamma(U, F)$ for $F(U)$, and call elements sections of $F$ over $U$.


Let $\Sigma$ be an affine alg. set, with the Zariski topology. Not all continuous maps $\Sigma_1 \to \Sigma_2$ are morphisms. What we need to do is consider not only the space but also a sheaf.

We will define a sheaf of rings $(\Sigma, O_\Sigma)$.

Def. Let $X \subseteq k^n$ be an irreducible alg. set, $R = \Gamma(X)$. Let $K = \frac{1}{R}$. For each $x \in X$, let $m_x \subseteq R$ be the corresponding maximal ideal, and let $O_{X,x} = R_{m_x} = \{f \in R : g(x) \neq 0\}$. Define $O_X(U) = \bigcap_{x \in U} R_{m_x} \subseteq K$. 
