1 Recap

We know now that products of prevarieties exist.

Here’s a note about the next homework:
Let $\mathcal{C}$ be a category, let $X \in \mathcal{C}$. Then we define $h_X : C \to \text{Set}$ defined by $h_X : Y \mapsto \text{Hom}_C(Y, X)$.

Yoneda’s Lemma. If $X, Y \in \text{Ob}(\mathcal{C})$ then there is a natural bijection $\text{Hom}_C(X, Y) \cong \text{Hom}(h_X, h_Y)$ (natural transformations of functors).

How does this relate back? Because $h_{X \times Y}(Z) = \text{Hom}(Z, X) \times \text{Hom}(Z, Y) = h_X(Z) \times h_Y(Z)$.

Def. A functor $F : C \to \text{Set}$ is representable if it is of the form $h_X$.
So our big result is basically that $h_X \times h_Y$ is representable.
A lot of abstract mumbling.

2 Next definition of variety

We want to figure out what makes a prevariety a variety. (Note: we are defining variety here, not affine variety or projective variety. They will of course agree, but that’s for a little later.)

Consider $\mathcal{X} = \mathbb{A}^1 \coprod_{\mathbb{A}^1 - \{0\}} \mathbb{A}^1$. We want this kind of thing NOT to be a variety.

Def./Prop. A prevariety $Y$ is a variety if the following equivalent conditions hold:

1. $\forall$ pairs of morphisms $f, g : X \to Y$ the set $\{x \in X | f(x) = g(x)\} \subset X$ is closed.\(^1\)

2. $\forall$ morphisms $f : X \to Y$, the graph $\Gamma_f : X \xrightarrow{1 \times f} X \times Y$ has a closed image.

3. The diagonal $\Delta : Y \xrightarrow{1 \times 1} Y \times Y$ has closed image.

Pf. that these are equivalent. (1) $\iff$ (3). To prove $\Rightarrow$, consider $X = Y \times Y$, where $f$ is projection from the first and $g$ is projection from the second part. From (1) we know that $\{(y_1, y_2) | p_1(y_1, y_2) = p_2(y_1, y_2)\}$ is closed, but $p_1(y_1, y_2) = y_1$ and $p_2(y_1, y_2) = y_2$ so this is just the image of $\Delta$.

To prove $\Leftarrow$, suppose we have $f, g : X \to Y$. Then we have diagram:

\[
\begin{array}{ccc}
X & \xleftarrow{(f \times g)^{-1}(\Delta(Y))} & Y \\
\downarrow f \times g & & \downarrow \\
Y \times Y & \xleftarrow{\Delta(Y)} &
\end{array}
\]

\(^1\)This fails in $\mathcal{X}$; let $X = \mathbb{A}^1$ and let $f$ be inclusion via the first set, $g$ via the second. These agree on $\mathbb{A}^1 - \{0\}$ under either inclusion which is not closed.
where $(f \times g)^{-1}(\Delta(Y))$ is just the set we’re interested in. We can see this is closed by the diagram.

(1) $\iff$ (2). To prove $\Rightarrow$, let $X' = X \times Y$, let $f' : (x, y) \mapsto f(x)$ and let $g' : (x, y) \mapsto y$. Then the set (1) proves is closed is just $\{(x, y)|y = f(x)\}$ which is what we want to prove.

2 $\Rightarrow$ 3 is easy: just use $f = 1$.

**Def.** A sub-prevariety of $X$ is locally closed subset with induced structure of a prevariety. (Remember, $U$ locally closed means there is some open set $V$ such that $U \subset V$ and $U$ is closed in $V$.)

**Lemma.** A sub-prevariety $Z$ of a variety $X$ is a variety.

**Pf.** Let’s check (3). We know $\Delta_X(X)$ is closed in $X \times X$. So we intersect this with $Z \times Z$ and we get $\Delta_Z(Z)$ and this is closed since the map $Z \times Z \mapsto X \times X$ is a morphism.

**Lemma.** A product of two varieties $X$ and $Y$ is a variety.

**Pf.** Consider $f \times g, f' \times g' : Z \mapsto X \times Y$. We are interested in the set $\{z \in Z|(f \times g)(z) = (f' \times g')(z)\} = \{z \in Z|f(z) = f'(z)\} \cap \{z \in Z|g(z) = g'(z)\}$ which are each closed sets since $X$ and $Y$ are varieties.

**Lemma.** An affine variety is a variety.

**Pf.** Check (3). Let $R = \Gamma(X, O_X)$, we know that $X \times X$ corresponds to $R \otimes_k R$ and $\Delta$ corresponds to a map $R \otimes_k R \xrightarrow{\Delta} R$ defined by $\Delta^* : (r \otimes s) \mapsto rs$. This is surjective, clearly. Let $J$ be the kernel. Now, $\Delta_X$ is just $V(J)$, so $\Delta(X)$ is closed.

**Lemma.** If $f, g : X \mapsto Y$ are two morphisms of prevarieties, then the set $E_{fg} = \{x \in X|f(x) = g(x)\}$ may not be closed, but is locally closed.

**Pf.** Say $x \in E$, and let $U \subset Y$ be an affine open set containing $f(y) = g(y)$. Note that $V = f^{-1}(U) \cap g^{-1}(U)$ which is an open set in $X$ containing $x$. Then $E \cap V = \{v \in V|f(v) = g(v)\}$. But now, $f, g : V \mapsto U$ where $U$ is affine. Thus, $E \cap V$ is closed in $V$.

To complete this, note that we can give $E$ a finite cover of the form $E \cap V$ where each is closed; thus, $E$ is locally closed (by quasi-compactness).

**Prop.** Let $X$ be a prevariety. Assume $\forall x, y \in X$ there is an open affine $U \subset X$ containing both $x, y$. Then $X$ is a variety.

**Cor.** A projective variety is a variety.

**Pf.** of Cor. Let $X \subset \mathbb{P}^n$. Say $x, y \in X$. Then there is a line which is not zero at either $x$ or $y$. Take $\mathbb{P}^n$ minus the zero-locations of this line, and we get an open affine set containing $x$ and $y$.

**Pf.** of Prop. Look at $f, g : Z \mapsto X$, consider $E_{fg}$. We can take the closure $E_{fg} \subset \overline{E_{fg}} \subset Z$. Now we choose $z \in E$. Let $x = f(z), y = g(z)$. Choose an affine open $U \subset X$
such that \( x, y \in U \). Now \( z \in f^{-1}(U) \cap g^{-1}(U) \to U \) where \( U \) is a variety. We also have \( E_{f,g} \cap f^{-1}(U) \cap g^{-1}(U) \subset f^{-1}(U) \cap g^{-1}(U) \) closed, so \( z \in E_{f,g} \). Thus, every point in the closure is actually in \( E_{f,g} \) so \( E_{f,g} \) must be closed.

So now we have consistency of definitions: affine varieties and projective varieties are kinds of varieties.

At this point we’re done with all the definitions, and now just want to study the properties of varieties.

3 Dimension

**Def.** Let \( X \) be a variety. The **dimension** of \( X \) is denoted \( \dim X \) and is the transcendence degree of \( k(X) \) over \( k \) (denoted \( \text{tr.deg}_k k(X) \)).

**Examples.**

- \( \mathbb{A}^n \) has dimension \( n \).
- \( \mathbb{P}^n \) has dimension \( n \) because it is covered by \( \mathbb{A}^n \)'s.
- If \( X \) has dimension 0, then \( X = (*, k) \), the variety of one element.

**Remark.** (A topological aside.) Let \( X \) be a topological space, and consider chains of irreducible prevarieties. We will see later that the dimension of \( X \) is the maximum length of the chain

\[
Z_0 \subset Z_1 \subset \ldots \subset Z_n = X
\]

where \( Z_i \subset X \) are irreducible closed prevarieties. The idea here is something like: \( \mathbb{A}^3 \) has dimension 3 because any closed prevariety in \( \mathbb{A}^3 \) is a planar manifold, and any closed prevariety on a planar manifold is a linear manifold, and any closed prevariety on a linear manifold is a point.

**Prop.** Say \( Y \subset X \) is a proper closed subvariety of \( X \). Then \( \dim Y < \dim X \).

**Pf.** We can assume \( X \) is affine, and let \( R = \Gamma(X, O_X) \). Since \( Y \) is irreducible, we know that \( Y \) corresponds to some prime ideal \( P \subset R \) and we can assume \( P \neq (0) \) or \( R \) since \( Y \) is proper.

To restate then, if we have a finitely generated \( k \)-algebra \( R \), and \( P \subset R \) a nontrivial prime ideal of \( R \), then \( \text{tr.deg} \frac{R}{P} < \text{tr.deg} \frac{R}{R} = n \).

Now we prove this. If not, then there exist \( x_1, \ldots, x_n \in R \) such that \( \overline{x_1}, \ldots, \overline{x_n} \) in \( R/P \) are algebraically independent. Choose \( 0 \neq p \in P \). Then, there is a polynomial \( F(Y, X_1, \ldots, X_n) \) such that \( F(p, x_1, \ldots, x_n) = 0 \). But then when we reduce this polynomial modulo \( P \) we get a contradiction since this will give a relation \( F'(x_1, \ldots, x_n) = 0 \). This completes the proof.