1 HW 5 problem

Thm. If $R$ and $S$ are integral domains and $k$-algebras, then $R \otimes_k S$ is an integral domain.

Pf. It is enough to consider $R$ and $S$ finitely generated, since we can write $R = \cup_i R_i$ and $S = \cup_j S_j$ so $R \otimes S = \cup_{i,j} R_i \otimes S_j$.

Say $f, g \in R \otimes S$ not zero but $fg = 0$. Then $f, g \in R_i \otimes S_j$ for some $i, j$.

Consider $k[x_1, \ldots, x_n] \rightarrow R$ and $k[y_1, \ldots, y_m] \rightarrow S$. Let $X \subset \mathbb{A}^n$ and $Y \subset \mathbb{A}^m$, and consider

$J \rightarrow k[x_1, \ldots, x_n, y_1, \ldots, y_m] \rightarrow R \otimes S$.

We will prove that $J$ is a prime ideal. To do this, we consider $Z \subset \mathbb{A}^{n+m}$ the closed algebraic set defined by $J$. We will prove that $Z$ is irreducible, and then that $\sqrt{J} = J$.

To show that $Z$ is irreducible, suppose $Z = Z_1 \cup Z_2$ where $Z_i \subset Z$ are closed. We have maps of topological spaces $Z \xrightarrow{p_1} X$ and $Z \xrightarrow{p_2} Y$. Observe: if $x \in X$ then $p_1^{-1}(x) \rightarrow Y$ is a homeomorphism, because $p^{-1}(x)$ is the set of quotients $R \otimes S \rightarrow k$ which factor through $R \otimes S \xrightarrow{x \otimes 1} S$.

So, $p_1^{-1}(x)$ is irreducible. Also, however, $p_1^{-1}(x) = (Z_1 \cap p_1^{-1}(x)) \cup (Z_2 \cap p_1^{-1}(x))$ so we know $p_1^{-1}(x) \subset Z_i$ for some $i$.

Define $X_i = \{ x \in X | p_1^{-1}(x) \subset Z_i \}$. We know $X = X_1 \cup X_2$. If we can prove that $X_i \subset X$ are closed, then $X = X_{i_0}$ for either $i_0 = 1$ or $2$, so $Z = Z_{i_0}$.

To prove that $X_i$ is closed, consider for every $y \in Y$, the set $X_i(y) = p_2^{-1}(x \cap p_1^{-1}(y)) = \{ x \in X | (x, y) \in Z_i \}$. Note that $X_i = \cap_y X_i(y)$. Also note that $X_i$ is the preimage of $Z_i$ under the map $x \rightarrow (x, y)$, so $X_i$ is closed, and therefore $X_i$ is closed.

Now to prove that $\sqrt{J} = J$, we just need to show there aren’t any nilpotent elements in $R \otimes S$. Say $h = \sum f_i \otimes g_i$ is nilpotent in $R \otimes S$ where $\{ f_i \}$ and $\{ g_i \}$ are linearly independent. For every $x \in X$ we get some $\sum f_i(x) g_i \in S$ which is nilpotent, which cannot be because $S$ is an integral domain. Thus, $f_i(x) = 0$ for every $x \in X$. Then, $h = 0$ because $f_i = 0$ for every $i$.

We use that $k$ is algebraically closed in order to map $R \otimes S \xrightarrow{x \otimes 1} S$ because this is essentially a quotient of $R$ by a maximal ideal, which we only know is $k$ if $k$ is algebraically closed (or it could be some algebraic extension of $k$).

2 Dimension

If $X$ is a variety, then $\dim(X) = \text{tr.deg}_k k(X)$. Last time we showed that if $Z \subset X$ is irreducible, proper, and closed, then $\dim(Z) < \dim(X)$.

Thm. If $X$ is a variety, $g \in \Gamma(X, O_X)$ non-zero, let $Z$ be an irreducible component of $\{ x \in X | g(x) = 0 \}$. Then $\dim(Z) = \dim(X) - 1$. So for example, a hypersurface in $\mathbb{P}^n$ (a variety defined by a single polynomial) has dimension $n - 1$.

It suffices to consider $X$ affine. We write $R = \Gamma(X, O_X)$. 

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The irreducible components of $V(g)$ correspond to the minimal primes $P \subset R$ that contain $g$.

**Thm.** (Krull’s principal ideal theorem): If $R$ is an integral domain, finitely generated $k$-algebra, $g \in R$ not zero, and $P$ is a minimal prime containing $g$, then $\text{tr.deg}_k R/P = \text{tr.deg}_k (R) - 1$.

**Pf.** of KPT. We want to reduce to the case $X = \mathbb{A}^n$.

Recall the Noether Normalization lemma: if $X$ is an affine variety of dimension $n$, then there is a finite surjective morphism $\pi : X \to \mathbb{A}^n$. Recall that finite here means that $\Gamma(X, O_X)$ is a finitely generated $k[x_1, \ldots, x_n]$-module. This all matches up with our previous way of stating Noether normalization.

**Lemma.** Let $f : X \to Y$ be a finite morphism. Then
1. $f$ is a closed map
2. Fibers are finite (from HW)
3. $f$ is surjective $\iff$ $S = \Gamma(Y, O_Y) \to R = \Gamma(X, O_X)$ is injective.

**Pf.** of lemma. (2) was from HW. (1) and (3). Say we have $V(A) \subset X$. What is $f(V(A)) \subset Y$? It is the set of maximal ideals $P^{-1}(m)$ where $m \subset R$ is maximal and contains $A$.

Let $P^{-1}(A) = B$ be an ideal in $S$. We claim: $f(V(A)) = V(B)$. Note we have $S/B \to R/A$ integral. By the going-up theorem, for every maximal ideal $n \subset S/B$ there is an $m \subset R/A$ such that $m \cap S/B = n$. This shows that $f$ is a closed map.

Now, if we take $A$ to be the zero ideal, then $f(X) = V(Ker(S \to R))$. This proves (3), because if $f$ is surjective then $f(X) = Y$ so the kernel is 0, and if the kernel is 0 then $V(0) = Y$ so $f$ is surjective. This completes the proof of the lemma.

Now back to our proof of KPT. We reduce to the case $P = \sqrt{(g)}$, which gets rid of all the components of $Z$ except one. We can write

$$\sqrt{(g)} = P \cap P_1 \cap \ldots \cap P_t.$$

Choose $f \in P_1 \cap \ldots \cap P_t$ such that $f \not\in P$ (by the prime avoidance theorem), and consider $D(f) \subset X$. Localize at $f$, and we will be left with the case $\sqrt{(g)} = P$.

**Remark.** If $R$ is a UFD then we’re done. This is because $g = eh^t$ where $h$ is irreducible and $P = (h)$ and $e$ a unit. Thus we want the transcendence degree of $R/(h)$ which is clearly $\text{tr.deg} R - 1$.

For the general case, choose a finite map $S = k[x_1, \ldots, x_n] \to R$. Let $L = \text{Frac} (S)$ and let $K = \text{Frac} (R)$. Let $g_0 = \text{Norm}_{K/L}(g)$. Recall that $\text{Norm}_{K/L} = \det(*g : K \to K)$ where we think of $*g$ the map (which multiplies by $g$) as a matrix. Then $g_0 \in S \cap P$. This is not obvious, so look it up.

Then we have $\text{tr.deg} R/P = \text{tr.deg} S/(S \cap P)$. Claim: $S \cap P = \sqrt{(g_0)}$. We have that $P \cap S \supset \sqrt{(g_0)}$. For the other way, say that $h \in P \cap S$. Then $h \in P$, so $h^t = fg$ in $R$. Now we take the norm of both sides and get $h^t[K:L]$ on the left, and $\text{Norm} (f)g_0$ on the right since the determinant is multiplicative, so a power of $h$ is in $(g_0)$. Now we can apply the same argument from when we had $R$ as a UFD, so this completes the proof of KPT.
3 Next stuff

**Def.** Let \( X \) be a variety, \( Z \subset X \) closed. Then \( Z \) has pure (co-)dimension \( r \) if all the irreducible components of \( Z \) have (co-)dimension \( r \).

**Cor.** If \( g \in \Gamma(X, O_X) \) non-zero then \( V(g) \subset X \) has pure co-dim 1.

**Remark.** Here is the converse. Let \( Z \subset X \) be irreducible with codim 1 and a nonzero \( g \in \Gamma(X, O_X) \) such that \( g(Z) = 0 \). Then \( Z \) is an irreducible component of \( V(g) \) (as opposed to a subset of an irreducible component).

**Cor.** Let \( X \) be a variety, \( Z \subsetneq X \) a maximal irreducible closed subset. Then \( \dim Z = \dim X - 1 \).

**Pf.** We can assume \( X \) is affine, so \( Z \) is associated with some ideal \( (f_1, \ldots, f_l) \subset \Gamma(X, O_X) \), and note that \( Z \subset V(f_1) \). So \( Z \) is equal to an irreducible component of \( V(f_1) \) so \( \dim Z = \dim X - 1 \).

**Cor.** Say \( \emptyset \neq Z_1 \subsetneq Z_2 \subsetneq \ldots \subsetneq Z_r \subsetneq X \) is a maximal length chain of irreducible closed subsets. Then \( \dim X = r \). (By induction.)