Last homework due Dec. 2, take home final due Dec. 10. Make-up lecture Dec. 11.
11/20/03

Let \( X \) be a variety. We had from last time a map
\[
O_X \xrightarrow{d} \Omega^1_X, \text{ a sheaf of } O_X\text{-modules, where } d \text{ was a derivation.}
\]

For all affine \( U \subset X, R = \Gamma(U, O_X) \), then there was a map \( R \xrightarrow{d} \Omega^1_X(U) \) was the universal derivation.

A derivation is a map \( \delta : R \rightarrow M \) such that \( \delta \) is \( k \)-linear and \( \delta(fg) = f \delta(g) + g \delta(f) \). \( d \) is universal such that \( R \xrightarrow{d} \Omega^1_X(U) \) factors any \( \delta : R \rightarrow M \) with a unique \( R \)-linear map from \( \Omega^1_X(U) \rightarrow M \).

Let \( R = k[X_1, \ldots, X_n]/(f_1, \ldots, f_r) \). Then \( \Omega = Rdx_1 \oplus \cdots \oplus Rdx_n/(df_i) \).

That is, \( \Omega \) is the cokernel of the map \( R^r \xrightarrow{J} Rdx_1 \oplus \cdots \oplus Rdx_n \) where \( J \) is the jacobian
\[
\begin{pmatrix}
\delta f_1/\delta x_1 & \cdots & \delta f_r/\delta x_1 \\
\vdots & & \vdots \\
\delta f_1/\delta x_n & \cdots & \delta f_r/\delta x_n
\end{pmatrix}
\]

where \( e_i \mapsto (\sum_j \frac{\delta f_i}{\delta x_j} dx_j) \).

Then \( \Omega^1_X(D(f)) = \Omega \otimes_R Rf \). If \( x \in X \) then the stalk \( \Omega^1_{X,x} = \Omega \otimes_R \mathcal{O}_{m_x} \).

We write \( \Omega^1_X(x) \) to be \( \Omega^1_{X,x} \otimes \mathcal{O}_{m_x} k \) from \( \Omega^1_{X,x} \otimes_R k \) via the quotient map \( R \rightarrow \mathcal{O}_{m_x} \).

**Example.** \( V(y^2 - x^2(x + 1)) \subset \mathbb{A}^2 \).

\[
R \xrightarrow{(-3x^2-2y, 2x)} Rdx \oplus Rdy \rightarrow \Omega.
\]

If we have \( x = (a, b) \), then we have the sequence
\[
k \xrightarrow{(-3a^2-2b, 2a)} dx \oplus dy \rightarrow \Omega^1_X(x) \rightarrow 0
\]

by tensoring with \( k \). Then \( \dim \Omega^1_X(x) = 2 \) if \( x = (0, 0) \), and \( 1 \) if \( x \neq (0, 0) \).

**Lemma.** Let \( X \) be a variety, \( x \in X \). Then there is a natural isomorphism \( \text{Hom}_{O_{X,x}}(\Omega^1_{X,x}, k) = \text{Hom}_k(\Omega^1_X(x), k) = \text{Hom}_k(m_x/m^2_x, k) \). This last equality is what we must prove; the first is really just from properties of tensor products.

**Cor.** \( O_{X,x} \) is regular \( \iff \dim_k \Omega^1_X(x) = \dim X \).

**Pf.** of Lemma. \( \text{Hom}_{O_{X,x}}(\Omega^1_{X,x}, k) \) is the set of derivations \( \delta : O_{X,x} \rightarrow k \). This is cheating a little because we only know this for affine opens, and this statement is passing to the limit, but it’s okay.

Now note: any such map must kill \( m^2_x \), because \( \delta(xy) = x\delta(y) + y\delta(x) \) so if both things are in \( m_x \) then both parts on the right are in \( m_x \) and are thus 0 in \( k \). So our set of homomorphisms is just the set of linear maps \( m/m^2 \rightarrow k \).

We have a map
\[
m/m^2 \rightarrow O_{X,x}/m^2 \xrightarrow{\delta} k,
\]
where the composition is \( k \)-linear. Note that for any \( f \in O_{X,x} \), we have that \( \delta(f) = \delta(f(x) + (f - f(x)) = \delta(f - f(x)) \) where \( f - f(x) \) is in the maximal ideal so it is determined by this map.

\textbf{Cor.} \ \ \ \dim_k \Omega^1_X(x) = \dim_k m/m^2.

\textbf{Def.} \ \ x \in X \ is \ a \ smooth \ point \ if \ \ O_{X,x} \ is \ regular. \ \ If \ x \ is \ not \ a \ smooth \ point \ is \ called \ a \ singular \ point. \ \ Basically \ singular \ points \ correspond \ to \ points \ where \ the \ Jacobian \ is \ "\text{wrong}" \ \ (ie, \ has \ strange \ rank \ or \ something).

\textbf{Def.} \ \ The \ tangent \ sheaf \ of \ \ X \ is \ \ coHom_{O_X} (\Omega^1_X, O_X).

\textbf{Aside.} \ \ If \ \ \mathcal{F}, \mathcal{G} \ are \ sheaves \ of \ \ O_X\text{-modules}, \ then \ \ coHom_{O_X} (\mathcal{F}, \mathcal{G}) \ is \ the \ sheaf\footnote{not \ obvious \ that \ it \ is \ a \ sheaf} \ \ \text{mapping} \ \ \mathcal{U} \mapsto Hom_{O_X(U)}(\mathcal{F}(U), \mathcal{G}(U)).

So \( T_{X,x} = Hom_{O_{X,x}} (\Omega^1_{X,x}, O_{X,x}) \).

\section{Curves}

\textbf{Def.} \ \ A \ curve \ is \ a \ variety \ of \ dimension \ 1.

This is rather abstract as we’ve done things.

\textbf{Theorem.} \ \ The \ functor \ from \ the \ category \ of \ complete \ smooth \ curves \ with \ non-constant \ morphisms \ to the \ category \ of \ finitely \ generated \ field \ extensions \ \( k \to K \) \ of \ transcendence \ degree \ 1, \ with \ \( k \)-algebra morphisms \ is \ defined \ as \ follows.

\( C \to k(C) \). \ \ Clearly \ \( k(C) \) has \ transcendence \ degree \ 1 \ since \ \( C \) has \ dimension \ 1 \ (this \ was \ our \ original \ definition \ of \ dimension).

\( \ \) (1) This \ functor \ is \ an \ equivalence \ of \ categories. \ Furthermore, \ (2) every \ complete \ smooth \ curve \ is \ projective.

\textbf{Sketch \ of \ proof.} \ \ If \ we \ have \ \( p \in C \), \ what \ do \ we \ know \ about \ \( O_{C,p} \)? \ It’s \ a \ DVR, \ thanks \ to \ smoothness \ and \ dimension 1. Also, \ it \ sits \ inside \ \( k(C) \). \ This \ will \ give \ a \ bijection \ between \ the \ points \ of \ \( C \) and \ DVRs \ in \ \( k(C) \). \ It \ turns \ out \ this \ captures \ everything; \ given \ a \ field, \ the \ curve \ will \ be \ the \ set \ of \ DVRs \ in \ the \ field

\textbf{Aside \ about \ Number \ Theory.} \ \ If \ \( K \) is \ a \ number \ field, \ then \ \( O_K \) \ is \ the \ integral \ closure \ of \ \( Z \), \ then \ every \ prime \ in \ \( O_K \) \ corresponds \ to \ a \ valuation \ ring \ in \ \( K \). \ For \ example, \ \( K = \mathbb{Q}_5 \), \ \( O_K = \mathbb{Z} \), \ then \ for \ every \ \( q \in K, v_p(q) = ord_p(q) \). \ If \ this \ makes \ any \ sense \ to \ anyone.

\textbf{Def.} \ \ If \ \( K \) is \ a \ field, \ \( G \) a \ totally \ ordered \ group, \ then \ a \ \text{valuation} \ is \ a \ map \ \( v : K - \{ 0 \} \to G \) \ such \ that \ \( v(xy) = v(x) + v(y) \) \ and \ \( v(x + y) \geq \min\{ v(x), v(y) \} \). \ For \ us, \ the \ key \ example \ will \ be \ \( K = k(C), p \in C, v_p(f) \) \ is \ the \ order \ of \ the \ zero \ / \ pole \ at \ \( p \) \ of \ \( f \). \ For \ instance, \ if \ \( C = \mathbb{A}^1 \), \ then \ \( k(C) = k(t) \). \ If \ \( f \in k(t) \) \ then \ \( f = \alpha_1 t^r + \alpha_2 t^{r+1} + \ldots \) \ for \ some \ \( r \), \ so \ \( v(f) = r \).

\( R_v = \{ x : v(x) \geq 0 \} \cup \{ 0 \} \) and \ \( m_v = \{ x | v(x) > 0 \} \); \ these \ make \ a \ local \ ring.
\( v \) is discrete if we can take \( G = \mathbb{Z} \). In general, if \( C \) is a smooth curve, \( p \in C \), then \( v_p : k(C) \setminus \{0\} \to \mathbb{Z}; v_p : O_{C,p} \setminus \{0\} \to \mathbb{Z}, \) where \( v_p(f) = \max_n\{f \in m_p^n\} \). \( v_p(f/g) = v_p(f) - v_p(g) \).

**Def.** If \( K \) is a field, \( A, B \subset K \) local rings, we say that \( B \) dominates \( A \) if \( A \subset B \) and \( m_A = A \cap m_B \).

**Theorem.** Valuation rings are exactly the maximal local rings in \( K \) with respect to domination. [fact from AM]. We do not prove this here.

Let \( k \to K \) be a field extension of transcendence degree 1. Let \( C_K \) be the set of DVRs in \( K \). Say \( U \subset C_K \) is open if it is cofinite or if \( U = \emptyset \). Define \( O_{C_K}(U) \) to be \( \cap_{v \in \mathfrak{U}} O_v \), where \( O_v \) refers to the DVR corresponding to the valuation \( v \). We can think of elements of \( O_v \) as functions, as \( f(v) \) is the image of \( f \) in \( O_v/m_v = k \).

Now we need to show that \( (C_K, O_{C_K}) \) is a smooth curve.

**Lemma.** Say \( f \in K \). Then \( \{v \in C_K | f \notin O_v \} \) is a finite set.

**Pf.** We have a fact: \( \{v \in C_K | f \notin O_v \} = \{v \in C_K | 1/f \in m_v \} \). We know that \( f \notin k \) or this set would be empty and we’d be done. Write \( g = 1/f \). Let \( B \) be the integral closure of \( k[g] \subset K \). \( B \) corresponds to a smooth curve \( C \) because every local ring is regular (this is from property of Dedekind domains...)

If \( g \in m_v \), we get \( m_v \subset B \to O_v \to \text{eval}k \) which corresponds to points in \( V(g) \subset C \), so it’s finite.

This also shows \( (C_K, O_{C_K}) \) is a variety. Say \( v \in C_K \). Choose \( g \in O_v \) non-zero. This corresponds to a smooth curve \( C, k(C) = K \), where \( \{v | R_v = O_{C,p}, p \in C \} \) is an open set around \( v \).

**Lemma.** Say \( Y \) is an affine variety, \( P, Q \in Y \) and say \( O_{Y,P} \subset O_{Y,Q} \subset k(Y) \). Then \( P = Q \). [Proof ommitted]

Out of time.