Let $Z$ be an irreducible closed subset in an algebraic variety $X$. Show that if $\dim(Z) = \dim(X)$ then $Z$ is a component of $X$.

Let $Y$ be a closed subvariety of dimension $r$ in $\mathbb{P}^n$.

(a) Suppose that $Y$ can be presented as the set of common zeroes of $q$ homogeneous polynomials. Show that $r \geq n - q$.

(b) Show that every irreducible closed subvariety in $\mathbb{P}^n$ is a component in a set theoretic complete intersection of the same dimension.

(c) Show that the twisted cubic curve in $\mathbb{P}^3$ (see problem 2 of problem set 2) is a set theoretic complete intersection.

(d) (Optional bonus problem) Show that the twisted cubic curve in $\mathbb{P}^3$ is not a strict complete intersection.

Let $C$ be a curve in $\mathbb{P}^2$, $x$ be a point in $C$ and $L$ a line passing through $x$. Let $m$ be the multiplicity of $C$ at $x$ and $M$ the multiplicity of intersection of $C$ and $L$ at $x$. Show that $m \leq M$ and that for given $C$, $x$ the equality $m = M$ holds for all but finitely many lines $L$ as above.

Prove Bezout Theorem for two curves of degrees $d_1$, $d_2$ in $\mathbb{P}^2$ with no common components.

(a) Assuming $d_1 = 1$.

(b) Assuming $d_1 = 2$ and the first curve is irreducible; you can also assume that characteristic of the base field is different from two.

(5) (Optional bonus problem) Recall from the lecture that Grassmannian $Gr(2, 4)$ is isomorphic to a quadric in $\mathbb{P}^5$. Use this to show that given four lines in $\mathbb{P}^3$, the number of lines intersecting each of the four lines is either infinite or equal to one or two.
[Hint: Check that for a line $L \subset \mathbb{P}^3$ the set of lines intersecting $L$ is parametrized by $Gr(2, 4) \cap H$ for a hyperplane $H \subset \mathbb{P}^5$, thus the answer is the number of points in the intersection $L \cap Gr(2, 4)$ where $L \subset \mathbb{P}^5$ is a linear subspace of dimension one or higher. Check that the intersection is infinite unless $L$ is a line and refer to problem 3(a) from problem set 2].